

Semistrict Higher Gauge Theory

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Abstract

We develop semistrict higher gauge theory from first principles. In particular, we describe the differential Deligne cohomology underlying semistrict principal 2-bundles with connective structures. Principal 2-bundles are obtained in terms of weak 2-functors from the Čech groupoid to weak Lie 2-groups. As is demonstrated, some of these Lie 2-groups can be differentiated to semistrict Lie 2-algebras by a method due to Ševera. We further derive the full description of connective structures on semistrict principal 2-bundles including the non-linear gauge transformations. As an application, we use a twistor construction to derive superconformal constraint equations in six dimensions for a non-Abelian $\mathcal{N} = (2, 0)$ tensor multiplet taking values in a semistrict Lie 2-algebra.

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1. Introduction, summary, and outlook

1.1. Motivation

Gauge theory is arguably one of the most important concepts in modern physics. For example, the impressive success of the standard model of elementary particles as well as many of the more recent developments in string theory are based on this concept. From a mathematical point of view, the kinematical data of classical gauge theory is described in terms of principal bundles with connection. Equivalence relations on this data, known as gauge transformations, are captured by a non-Abelian generalisation of the so-called first Deligne cohomology group.

By now, there is a well-established way of categorifying gauge theory to what is known as higher gauge theory. Here, the kinematical data lives on non-Abelian gerbes [1,2] or the more general principal 2-bundles of Bartels [3]; higher categorifications leading to p -gerbes or principal $p+1$ bundles are known, as well. The notion of a connection is generalised to a connective structure on a principal 2-bundle, which is also well established albeit with one limitation: instead of featuring the most general, weak Lie 2-group as structure 2-group, the standard formulations employ crossed modules of Lie groups, which are equivalent to strict Lie 2-groups.

The main aim of this paper is to lift this limitation and to discuss principal 2-bundles with connective structures that have semistrict Lie 2-groups as structure 2-groups. This involves considerably more technical effort than the strict case, and we would therefore like to give ample motivation for developing the detailed description of semistrict principal 2-bundles with connective structure.

The most general notion of a categorified group or 2-group which we shall consider here is what is usually called a weak 2-group. Just as a group is a groupoid with a single object, a weak 2-group is a bigroupoid with a single object. As shown by Baez & Lauda [4], every weak 2-group can be enhanced to a coherent 2-group, and, furthermore, coherent 2-groups are categorically equivalent to strict 2-groups. Categorical equivalence, however, seems to be too coarse in many cases. Perhaps a prime example in this regard is the categorified operation of integrating a Lie algebra: it is known that the string Lie 2-algebra cannot be integrated to a topological 2-group [4]. This semistrict Lie 2-algebra, however, is categorically equivalent to an infinite dimensional strict Lie 2-algebra, which can be integrated to a strict Lie 2-group [5]. Similarly, we expect that interesting dynamical models of connective structures on principal 2-bundles will not necessarily agree even if the underlying structure 2-groups are categorically equivalent.

Our motivation for considering categorified differential geometry stems mostly from M-theory, where principal 2-bundles with connective structures arise naturally in a non-Abelian generalisation of the effective description of M5-branes. In particular, they capture the kinematical structure of the $\mathcal{N} = (2, 0)$ superconformal six-dimensional field theory, or $(2, 0)$ -theory for short.

The existence of the $(2, 0)$ -theory has been shown by Witten [6], but it remains unclear if it should have a classical description in terms of a Lagrangian and equations of motion. Recently, however, with the success of M2-brane models (which, contrary to popular belief, proved to have Lagrangian formulation), various directions of research have been pursued to try to arrive at such a classical description. In fact, much of the current research in string theory is devoted to a more detailed understanding of the $(2, 0)$ -theory.

Since the Abelian six-dimensional tensor multiplet contains a 2-form gauge potential described by a $U(1)$ -gerbe, it is only natural to expect that the non-Abelian case is described by the connective structure of a principal 2-bundle. Principal 2-bundles with connective structures allow for the parallel transport of one-dimensional objects, which is certainly relevant in the description of the self-dual strings that form the boundaries of the M2-branes mediating M5-brane interactions. A detailed explanation of the higher gauge theory approach to M5-branes can be found in Fiorenza, Sati & Schreiber [7].

Besides its mathematical appeal, an important argument for the use of higher gauge theory is that principal 2-bundles can indeed yield superspace constraint equations for the $\mathcal{N} = (2, 0)$ superconformal tensor multiplet in six dimensions. This was shown in Saemann & Wolf [8, 9], and the derivation of these equations involved a description of the on-shell tensor multiplet in terms of certain holomorphic principal 2- and 3-bundles over a twistor space. Interestingly, a twistorial description might also give rise to a Lagrangian formulation of the theory, as was shown in the Abelian case in [10, 11].

The constraint equations arising from a twistorial description starting from principal n -bundles with strict Lie 2-groups turned out to be very restrictive. A first reason for considering semistrict principal 2-bundles is therefore to generalise the superconformal constraint equations arising from a twistor description of the $(2, 0)$ -theory and we shall present the outcome in Section 6.

Another popular approach to deriving a classical description of the $(2, 0)$ -theory is based on a non-Abelian generalisation of the tensor hierarchy [12] with the closely related proposals of [13]. Here, one obtains $\mathcal{N} = (1, 0)$ superconformal equations of motion as well as a Lagrangian description. These $(1, 0)$ -models have an underlying gauge algebraic structure, which is strongly reminiscent of a semistrict Lie 3-algebra. The detailed analysis of the this algebraic structures in [14] showed that there is indeed a large overlap. Moreover,

it was shown that certain classes of (1,0)-models are reformulations of higher gauge theories with strict Lie 3-groups. To fully compare the (1,0)-models with higher gauge theory, however, it is necessary to have a detailed description of gauge theory with semistrict principal n -bundles. This is a second motivation for studying semistrict principal 2-bundles.

Further motivation for our study stems from the problem of differentiating semistrict Lie 2-groups to semistrict Lie 2-algebras. While there has been some effort to understand the integration of Lie 2-algebras to Lie 2-groups, see for example Getzler [15] and Henriques [16], the inverse operation does not seem to have attracted the same amount of attention. In the present work, we shall follow a general approach to this problem that was proposed by Ševera [17]. In this approach, one considers a simplicial manifold and extracts a corresponding L_∞ -algebra as its first jet. A Lie 2-group can be encoded in a simplicial manifold as the Duskin nerve of its delooping. The first jet of this simplicial manifold is then constructed as a functor acting on descent data of a trivial principal 2-bundle.

Finally, note that a first proposal for semistrict higher gauge theory was given by Zucchini [18]. In this paper, the higher Maurer–Cartan forms were incorporated abstractly as constrained parameters into the gauge transformation. With our detailed understanding of the differential cohomology underlying semistrict principal 2-bundles with connective structure, however, we can make the parameters of gauge transformations explicit.

1.2. Summary of our results

For the reader’s convenience, let us summarise our key results in an easily accessible way. In the following, we fix a manifold X with covering $\mathfrak{U} := \{U_a\}$. Moreover, let $\mathbf{B}\mathcal{G} = (\{e\}, M, N)$ be a weak Lie 2-group which is a bigroupoid with a single 0-cell. Vertical and horizontal composition in this bigroupoid are denoted by \circ and \otimes , respectively.

A weak principal 2-bundle is described by a Čech 2-cocycle with values in $\mathbf{B}\mathcal{G}$. Such a cocycle is given by an N -valued Čech 2-cochain $\{n_{abc}\}$ together with an M -valued Čech 1-cochain $\{m_{ab}\}$, which satisfy the following semistrict cocycle conditions, cf. Definition 3.8:

$$\begin{aligned} n_{abc} : m_{ab} \otimes m_{bc} &\Rightarrow m_{ac} , \\ n_{acd} \circ (n_{abc} \otimes \mathrm{id}_{m_{cd}}) &= n_{abd} \circ (\mathrm{id}_{m_{ab}} \otimes n_{bcd}) \circ \mathfrak{a}_{m_{ab}, m_{bc}, m_{cd}} . \end{aligned} \tag{1.1}$$

Two principal 2-bundles are called equivalent if and only if their degree-2 Čech cocycles are related by a Čech 2-coboundary with values in $\mathbf{B}\mathcal{G}$. This coboundary consists of an N -valued Čech 1-cochain $\{n_{ab}\}$ and an M -valued Čech 0-cochain $\{m_a\}$ such that for degree-2

Čech cocycles $(\{n_{abc}\}, \{m_{ab}\})$ and $(\{\tilde{n}_{abc}\}, \{\tilde{m}_{ab}\})$ the following holds, cf. Definition 3.10:

$$\begin{aligned} n_{ab} : m_{ab} \otimes m_b &\Rightarrow m_a \otimes \tilde{m}_{ab} , \\ n_{ac} \circ (n_{abc} \otimes \text{id}_{m_c}) &= (\text{id}_{m_a} \otimes \tilde{n}_{abc}) \circ \mathbf{a}_{m_a, \tilde{m}_{ab}, \tilde{m}_{bc}} \circ (n_{ab} \otimes \text{id}_{\tilde{m}_{bc}}) \circ \\ &\quad \circ \mathbf{a}_{m_{ab}, m_b, \tilde{m}_{bc}}^{-1} \circ (\text{id}_{m_{ab}} \otimes n_{bc}) \circ \mathbf{a}_{m_{ab}, m_{bc}, m_c} . \end{aligned} \quad (1.2)$$

Consider now a semistrict Lie 2-group $\mathcal{B}\mathcal{G}$, which is a weak Lie 2-group in which left- and right-unitors as well as the units and counits are all trivial. There is a functor from the category of smooth manifolds to the category of descent data for weak principal 2-bundles on surjective submersions $\mathbb{R}^{0|1} \times X \rightarrow X$. This functor is parameterised by a 2-term L_∞ -algebra $L = (\mathfrak{m}, \mathfrak{n})$ as shown in Theorem 4.24. This 2-term L_∞ -algebra is the semistrict Lie 2-algebra corresponding to the semistrict Lie 2-group $\mathcal{B}\mathcal{G}$. Deriving the parameterisation of the functor is the higher equivalent of computing the Lie algebra of a Lie group.

Local connective structures on principal 2-bundles with semistrict structure 2-group (as well as principal n -bundles with semistrict structure n -group) are readily derived. One simply considers the tensor product of the 2-term L_∞ -algebra corresponding to the structure 2-group with the differential graded algebra of differential forms on a local patch U of the manifold X . This leads to another L_∞ -algebra, and its homotopy Maurer–Cartan equations as well as their infinitesimal symmetries yield the definition of a (flat) connective structure as well as infinitesimal gauge transformations as shown in Proposition 5.3.

The finite gauge transformations are derived by considering the equivalence relation on the functors considered in the above differentiation of Lie 2-groups $\mathcal{B}\mathcal{G} = (\{e\}, M, N)$ to 2-term L_∞ -algebras $L = (\mathfrak{m}, \mathfrak{n})$. This relation is presented in Theorem 4.26, from which Proposition 5.11 can be gleaned. A connective structure on a semistrict principal 2-bundle is given over U_a in terms of a differential 2-form $B_a \in H^0(U_a, \Omega_X^2 \otimes \mathfrak{n})$ and a differential 1-form $A_a \in H^0(U_a, \Omega_X^1 \otimes \mathfrak{m})$. Gauge transformations are parameterised by elements $p_a \in H^0(U_a, M)$ and $\Lambda_{p_a} \in H^0(U_a, \Omega_X^1 \otimes T_{p_a} N)$ as follows:

$$\Lambda_{p_a} : \tilde{A}_a \otimes p_a \Rightarrow p_a \otimes A_a - \text{d}p_a , \quad (1.3a)$$

$$\begin{aligned} \tilde{B}_a \otimes \text{id}_{p_a} &= \mu(\tilde{A}_a, \tilde{A}_a, p_a) + [\text{id}_{p_a} \otimes B_a + \mu(p_a, A_a, A_a)] \circ \\ &\quad \circ [-\text{d}\Lambda_{p_a} - \Lambda_{p_a} \otimes \text{id}_{A_a} - \mu(\tilde{A}_a, p_a, A_a)] \circ \\ &\quad \circ [-\text{id}_{\mathfrak{s}(\text{d}\Lambda_{p_a})} - \text{id}_{\tilde{A}_a} \otimes (\Lambda_{p_a} + \text{id}_{\text{d}p_a})] . \end{aligned} \quad (1.3b)$$

We can now combine our findings on Čech cohomology with values in a weak Lie 2-group with the form of finite gauge transformations of a local connective structure to the full semistrict Deligne cohomology of degree 2. The cocycle and coboundary relations are concisely listed in Definitions 5.18 and 5.19.

As a first application of our results, we employ semistrict Deligne cohomology of degree 2 in a twistor description of the $\mathcal{N} = (2, 0)$ superconformal tensor multiplet equations in six dimensions. This is a generalisation of previous results of [8, 9] from strict to semistrict gauge 2-groups. The main result here is Theorem 6.5, in which a bijection between certain holomorphic semistrict principal 2-bundles on a twistor space and the solutions to certain superconformal tensor multiplet equations in six-dimensions is established. We hope that the latter equations may serve as inspiration for a classical formulation of the $(2, 0)$ -theory.

1.3. Outlook

There is a number of questions arising from this paper, which we plan to address in future work. First of all, there should be an integration operation, inverse to our differentiation of a Lie 2-group to a semistrict Lie 2-algebra. An obvious question is how this integration is related to that of Getzler [15] and Henriques [16]. The answer seems to be similar to that found in [19] for the strict case. Here, straightforward Lie integration of a strict Lie 2-algebra led to a Lie 2-group which is Morita equivalent to the 2-group obtained by the method of Getzler and Henriques.

As mentioned above, we hope that the detailed description of semistrict principal 2-bundles with connective structure allows for a more detailed understanding of the framework of higher gauge theory. More general such theories can be considered and the relation to alternative approaches such as the non-Abelian tensor hierarchies should become clearer.

The most interesting dynamical theories involving connective structures on semistrict principal 2-bundles are certainly the $(2, 0)$ -theory and its dimensional reductions. As usual in supersymmetric physical theories, particular attention should be paid to the BPS subsector of these theories. Higher analogues of instantons and monopoles, as e.g. the self-dual strings, should be analysed from a mathematical perspective and especially the relevant topological invariants should be studied in detail. Some preliminary comments on this are given in [20]. General considerations concerning topological invariants in higher gauge theory can be found in [21] as well as in [22] from the perspective of Q -manifolds.

An important issue is to couple matter fields satisfyingly to higher gauge theories. In mathematical language, we would like to consider 2-vector bundles associated to our semistrict principal 2-bundles. Again, Zucchini [18] has suggested such a coupling; however, the existence of the gauge rectifiers necessary in this approach could not be proved so far. Our twistor construction gives a illuminating insight into how such a coupling should be achieved. It contains the explicit example of the matter fields contained in the tensor multiplet, what properties they satisfy, how gauge transformations act on them and how

they couple to the connective structure of the principal 2-bundle.

The most important consistency test for a classical (2,0)-theory is to reproduce five-dimensional maximally supersymmetric Yang–Mills theory in a certain limit. This is a requirement from string theory and so far, this has neither been achieved for higher gauge theories nor for the models arising from tensor hierarchies.

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2. Preliminaries

In this paper, we require basics of weak 2-category theory. We shall try to be as self-contained as possible and therefore we present the relevant definitions together with some useful examples in this section.

2.1. Weak 2-categories

We assume that the reader is familiar with elementary category theory. In the following, let $\mathcal{C} = (C_0, C_1)$ be a category with C_0 the objects of \mathcal{C} and C_1 the morphisms of \mathcal{C} , respectively. In addition, the source and target maps in \mathcal{C} are denoted by \mathbf{s} and \mathbf{t} , that is, $\mathbf{s}, \mathbf{t} : C_1 \rightarrow C_0$. We shall also use the standard notation $\mathcal{C} = (C_1 \rightrightarrows C_0)$ in which the double arrows refer to the source and target maps.

In higher category theory, there is always an issue concerning the level of strictness of the categorification under consideration. For example, 2-categories usually refer to strict 2-categories while weak 2-categories are often called bicategories. We shall exclusively use the terms weak 2-category, weak 2-groupoid etc. and avoid the notions of bicategory, bigroupoid etc. We shall start off with the definition of a weak 2-category. The original definition stems from Benabou [23], and a good introduction to the topic can be found, for instance, in [24] and in particular in [25]. The following discussion follows mostly these references.

Definition 2.1. (Benabou [23]) A weak 2-category $\mathcal{B} = (B_0, B_1, B_2)$ consists of a collection B_0 of objects $a, b, \dots \in B_0$ and, for any pair of objects $a, b \in B_0$, an assignment $(a, b) \rightarrow \mathcal{C}(a, b)$ where $\mathcal{C}(a, b) = (C_0(a, b), C_1(a, b))$ is a category. The objects B_0 are called

0-cells, the objects $C_0(a, b)$ are called 1-cells or 1-morphisms, and the morphisms $C_1(a, b)$ are called 2-cells or 2-morphisms.

In addition, \mathcal{B} comes equipped with a bifunctor $\otimes : \mathcal{C}(a, b) \times \mathcal{C}(b, c) \rightarrow \mathcal{C}(a, c)$ for all $a, b, c \in B_0$ describing horizontal composition¹ in \mathcal{B} , a functor² $\text{id} : 1 \mapsto \text{id}_a \in C_0(a, a)$ for all $a \in B_0$, and natural isomorphisms \mathbf{a} , \mathbf{l} , and \mathbf{r} defined by the commutative diagrams

$$\begin{array}{ccc} \mathcal{C}(a, b) \times \mathcal{C}(b, c) \times \mathcal{C}(c, d) & \xrightarrow{\otimes \times 1} & \mathcal{C}(a, c) \times \mathcal{C}(c, d) \\ \downarrow 1 \times \otimes & \nearrow \mathbf{a} & \downarrow \otimes \\ \mathcal{C}(a, b) \times \mathcal{C}(b, d) & \xrightarrow{\otimes} & \mathcal{C}(a, d) \end{array} \quad (2.1a)$$

and

$$\begin{array}{ccc} \mathcal{C}(a, b) \times 1 & & 1 \times \mathcal{C}(a, b) \\ \downarrow 1 \times \text{id} & \nearrow \mathbf{l} & \downarrow \text{id} \times 1 \\ \mathcal{C}(a, b) \times \mathcal{C}(b, b) & \xrightarrow{\otimes} & \mathcal{C}(a, b) \end{array} \quad \begin{array}{ccc} & \nearrow \cong & \\ & \searrow \cong & \\ & \xrightarrow{\otimes} & \mathcal{C}(a, b) \end{array} \quad (2.1b)$$

Here, the 1 attached to the arrows refers to the identity functor and \cong denotes the natural isomorphisms $1 \times \mathcal{C}(a, b) \cong \mathcal{C}(a, b) \cong \mathcal{C}(a, b) \times 1$. The natural isomorphisms \mathbf{a} , \mathbf{l} , and \mathbf{r} are referred to as the associator, left unitor, and right unitor, and they yield the 2-cells

$$\mathbf{a} : (x \otimes y) \otimes z \xrightarrow{\cong} x \otimes (y \otimes z), \quad \mathbf{l} : x \otimes \text{id}_b \xrightarrow{\cong} x, \quad \mathbf{r} : \text{id}_a \otimes x \xrightarrow{\cong} x \quad (2.2)$$

for $x \in C_0(a, b)$, $y \in C_0(b, c)$, and $z \in C_0(c, d)$. These isomorphisms are required to satisfy the pentagon and triangle identities, that is,

$$\begin{array}{ccc} ((x \otimes y) \otimes u) \otimes v & \xrightarrow{\mathbf{a} \otimes \text{id}} & (x \otimes (y \otimes u)) \otimes v \\ \downarrow \mathbf{a} & & \downarrow \mathbf{a} \\ (x \otimes y) \otimes (u \otimes v) & \xrightarrow{\mathbf{a}} x \otimes (y \otimes (u \otimes v)) \xleftarrow{\text{id} \otimes \mathbf{a}} & x \otimes ((y \otimes u) \otimes v) \end{array} \quad (2.3a)$$

and

$$\begin{array}{ccc} (x \otimes \text{id}_b) \otimes y & \xrightarrow{\mathbf{a}} & x \otimes (\text{id}_b \otimes y) \\ \searrow \mathbf{l} \otimes \text{id} & & \swarrow \text{id} \otimes \mathbf{r} \\ & x \otimes y & \end{array} \quad (2.3b)$$

are commutative.

¹as opposed to the composition of morphisms in $\mathcal{C}(a, b)$ which is referred to as vertical composition

²Here, the 1 is the terminal object in the category \mathbf{Cat} , that is, the singleton category consisting of one object e and the corresponding morphism id_e .

Remark 2.2. The fact that \otimes is a bifunctor implies the so-called interchange law, that is, the diagram

$$\begin{array}{ccccc}
 & x_1 & & y_1 & \\
 & \swarrow & & \swarrow & \\
 a & \xleftarrow{x_2} & b & \xleftarrow{y_2} & c \\
 & \searrow & & \searrow & \\
 & x_3 & & y_3 &
 \end{array}
 \begin{array}{c}
 \Downarrow f_1 \\
 \Downarrow f_2 \\
 \Downarrow g_1 \\
 \Downarrow g_2
 \end{array}
 \quad (2.4)$$

is commutative for $x_{1,2,3} \in C_0(a, b)$, $y_{1,2,3} \in C_0(b, c)$ and $f_{1,2} \in C_1(a, b)$, $g_{1,2} \in C_1(b, c)$ and $a, b, c \in B_0$. Equivalently, we may write

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1), \quad (2.5)$$

where \circ denotes the composition of 2-morphisms.

Remark 2.3. The naturalness of the associator \mathbf{a} implies that diagrams of the form

$$\begin{array}{ccc}
 (x \otimes y) \otimes z & \xrightarrow{(f \otimes g) \otimes h} & (f(x) \otimes g(y)) \otimes h(z) \\
 \mathbf{a} \downarrow & & \downarrow \mathbf{a} \\
 x \otimes (y \otimes z) & \xrightarrow{f \otimes (g \otimes h)} & f(x) \otimes (g(y) \otimes h(z))
 \end{array} \quad (2.6)$$

are commutative. There are similar commutative diagrams involving the unitors or a combination of unitors and associators.

Definition 2.4. A strict 2-category is a weak 2-category for which the associator and the left- and right-unitors are all trivial.

Example 2.5. The standard example of a strict 2-category is \mathbf{Cat} , regarded as a 2-category. Here, the 0-cells are given by small categories, the 1-cells are functors between those and the 2-cells are natural transformations between the latter.

Definition 2.6. A weak 2-category with a single 0-cell can be identified with a (weak) monoidal category. If, in addition, the natural isomorphisms \mathbf{a} , \mathbf{l} , and \mathbf{r} are all trivial, then we find a strict monoidal category.

The process of identifying n -categories with a single object or 0-cell with $(n - 1)$ -categories is called looping. Below, we shall also encounter the inverse operation called delooping.

Example 2.7. An example of a strict monoidal category is the category of sets endowed with a monoidal product given either by the Cartesian product or the disjoint union of sets. Here, $B_0 = \{e\}$ and $\mathcal{C}(e, e)$ is the category \mathbf{Set} whose objects C_0 are sets and whose morphisms C_1 are functions between sets.

In weak 2-categories with a single 0-cell, that is, in weak monoidal categories, we have the following result.

Proposition 2.8. (Kelly [26]) *In a weak monoidal category \mathcal{B} , the diagrams*

$$\begin{array}{ccc} (x \otimes y) \otimes \text{id}_c & \xrightarrow{a} & x \otimes (y \otimes \text{id}_c) \\ & \searrow \text{l} \quad \swarrow \text{id} \otimes \text{l} & \\ & x \otimes y & \end{array} \quad (2.7a)$$

$$\begin{array}{ccc} (\text{id}_a \otimes x) \otimes y & \xrightarrow{a} & \text{id}_a \otimes (x \otimes y) \\ & \searrow r \otimes \text{id} \quad \swarrow r & \\ & x \otimes y & \end{array} \quad (2.7b)$$

$$\begin{array}{ccc} & \text{l} & \\ \text{id}_a \otimes \text{id}_a & \xrightarrow{\quad} & \text{id}_a \\ & r & \end{array} \quad (2.7c)$$

are commutative for any $a, b, c \in B_0$ and $x \in C_0(a, b)$ and $y \in C_0(b, c)$.

Morphisms between categories are called functors. Similarly, morphisms between 2-categories are called 2-functors. These come in a number of variants, the most general of which are the lax 2-functors.

Definition 2.9. Let \mathcal{B} and $\tilde{\mathcal{B}}$ be two weak 2-categories. A lax 2-functor $\Phi : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$ is a triple $\Phi = (\Phi_0, \Phi_1, \Phi_2)$ consisting of a function $\Phi_0 : B_0 \rightarrow \tilde{B}_0$, a collection Φ_1 of functors

$$\Phi_1^{ab} : \mathcal{C}(a, b) \rightarrow \tilde{\mathcal{C}}(\Phi_0(a), \Phi_0(b)) , \quad (2.8a)$$

and a collection Φ_2 of 2-cells,

$$\begin{aligned} \Phi_2^{abc} : \Phi_1^{ab}(x) \tilde{\otimes} \Phi_1^{bc}(y) &\Rightarrow \Phi_1^{ac}(x \otimes y) , \\ \Phi_2^a : \text{id}_{\Phi_0(a)} &\Rightarrow \Phi_1^{aa}(\text{id}_a) , \end{aligned} \quad (2.8b)$$

where $a, b, c \in B_0$ and $x \in C_0(a, b)$ and $y \in C_0(b, c)$ such that the following diagrams are commutative:

$$\begin{array}{ccc} & \Phi_1^{ac}(x \otimes y) \tilde{\otimes} \Phi_1^{cd}(z) & \\ \nearrow \Phi_2^{abc} \otimes \text{id} & & \searrow \Phi_2^{acd} \\ (\Phi_1^{ab}(x) \tilde{\otimes} \Phi_1^{bc}(y)) \tilde{\otimes} \Phi_1^{cd}(z) & & \Phi_1^{ad}((x \otimes y) \otimes z) \\ \downarrow \tilde{a} & & \downarrow \Phi_1^{ad}(a) \\ \Phi_1^{ab}(x) \tilde{\otimes} (\Phi_1^{bc}(y) \tilde{\otimes} \Phi_1^{cd}(z)) & & \Phi_1^{ad}(x \otimes (y \otimes z)) \\ \searrow \text{id} \otimes \Phi_2^{bcd} & & \nearrow \Phi_2^{acd} \\ & \Phi_1^{ac}(x) \tilde{\otimes} \Phi_1^{cd}(y \otimes z) & \end{array} \quad (2.9a)$$

and

$$\begin{array}{ccccc}
& & \Phi_1^{ab}(x) \tilde{\otimes} \Phi_1^{bb}(\text{id}_b) & & \\
& \nearrow \text{id} \otimes \Phi_2^b & & \searrow \Phi_2^{abb} & \\
\Phi_1^{ab}(x) \tilde{\otimes} \text{id}_{\Phi_0(b)} & & & & \Phi_1^{ab}(x \otimes \text{id}_b) \\
& \searrow \tilde{r} & & \nearrow \Phi_1^{ab}(r) & \\
& & \Phi_1^{ab}(x) & & \\
& \nearrow \tilde{l} & & \searrow \Phi_1^{ab}(l) & \\
\text{id}_{\Phi_0(a)} \tilde{\otimes} \Phi_1^{ab}(x) & & & & \Phi_1^{ab}(\text{id}_a \otimes x) \\
& \searrow \Phi_2^a \otimes \text{id} & & \nearrow \Phi_2^{aab} & \\
& & \Phi_1^{aa}(\text{id}_a) \tilde{\otimes} \Phi_1^{ab}(x) & &
\end{array} \tag{2.9b}$$

Definition 2.10. A lax 2-functor $\Phi = (\Phi_0, \Phi_1, \Phi_2)$ for which the 2-cells Φ_2 are natural isomorphisms is called a weak 2-functor.³ A lax 2-functor $\Phi = (\Phi_0, \Phi_1, \Phi_2)$ for which the 2-cells Φ_2 are trivial is called a strict 2-functor.

Remark 2.11. Given two lax 2-functors $\Phi = (\Phi_0, \Phi_1, \Phi_2) : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$ and $\Psi = (\Psi_0, \Psi_1, \Psi_2) : \tilde{\mathcal{B}} \rightarrow \hat{\mathcal{B}}$, their composition $\Phi \circ \Psi$ yields another lax 2-functor $\Xi = (\Xi_0, \Xi_1, \Xi_2)$ with

$$\begin{aligned}
\Xi_0 &= \Psi_0 \circ \Phi_0 : B_0 \rightarrow \hat{B}_0 , \\
\Xi_1 &= \Psi_1^{\tilde{a}\tilde{b}} \circ \Phi_1^{ab} : \mathcal{C}(a, b) \rightarrow \hat{\mathcal{C}}(\Xi_0(a), \Xi_0(b)) , \\
\Xi_2^{abc} &= \Psi_1^{\tilde{a}\tilde{b}}(\Phi_2^{abc}) \circ \Psi_2^{\tilde{a}\tilde{b}\tilde{c}} : \Xi_1^{ab}(x) \tilde{\otimes} \Xi_1^{bc}(y) \Rightarrow \Xi_1^{ac}(x \otimes y) , \\
\Xi_2^a &= \Psi_1^{\tilde{a}\tilde{a}}(\Phi_2^a) \circ \Psi_2^{\tilde{a}} : \text{id}_{\Xi_0(a)} \Rightarrow \Xi_1^{aa}(\text{id}_a) ,
\end{aligned} \tag{2.10}$$

where $a, b, c \in B_0$ and $\tilde{a} = \Phi_0(a)$ etc.

Morphisms between functors are called natural transformations, and, as we shall see later on, they will become important when defining coboundary conditions. Therefore, we would like to introduce them in full detail.

Definition 2.12. Let $\Phi, \Psi : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$ be two lax 2-functors between two weak 2-categories \mathcal{B} and $\tilde{\mathcal{B}}$. A lax natural 2-transformation $\alpha : \Phi \Rightarrow \Psi$ with $\alpha = (\alpha_1, \alpha_2)$ consists of a family α_1 of 1-cells such that for each $a \in B_0$ there is a 1-cell $\alpha_1^a : \Phi_0(a) \rightarrow \Psi_0(a)$ and a family α_2 of 2-cells such that for each 1-cell $x \in C_0(a, b)$ in \mathcal{B} there is a 2-cell α_2^{ab} defined by

$$\begin{array}{ccc}
\Phi_0(b) & \xrightarrow{\Phi_1^{ab}(x)} & \Phi_0(a) \\
\alpha_1^b \downarrow & \nearrow \alpha_2^{ab}(x) & \downarrow \alpha_1^a \\
\Psi_0(b) & \xrightarrow{\Psi_1^{ab}(x)} & \Psi_0(a)
\end{array} \tag{2.11}$$

³Weak 2-functors are also known as pseudo-functors.

That is, $\alpha_2^{ab} : \Psi_1^{ab}(x) \tilde{\otimes} \alpha_1^b \Rightarrow \alpha_1^a \tilde{\otimes} \Phi_1^{ab}(x)$ such that for all $x \in C_0(a, b)$, $y \in C_0(b, c)$ and $a, b, c \in B_0$ the diagrams

$$\begin{array}{ccc}
\Psi_1^{ab}(x) \tilde{\otimes} (\alpha_1^b \tilde{\otimes} \Phi_1^{bc}(y)) & \xrightarrow{\tilde{a}^{-1}} & (\Psi_1^{ab}(x) \tilde{\otimes} \alpha_1^b) \tilde{\otimes} \Phi_1^{bc}(y) \xrightarrow{\alpha_2^{ab} \tilde{\otimes} \text{id}} (\alpha_1^a \tilde{\otimes} \Phi_1^{ab}(x)) \tilde{\otimes} \Phi_1^{bc}(y) \\
\uparrow \text{id} \tilde{\otimes} \alpha_2^{bc} & & \downarrow \tilde{a} \\
\Psi_1^{ab}(x) \tilde{\otimes} (\Psi_1^{bc}(y) \tilde{\otimes} \alpha_1^c) & & \alpha_1^a \tilde{\otimes} (\Phi_1^{ab}(x) \tilde{\otimes} \Phi_1^{bc}(y)) \\
\uparrow \tilde{a} & & \downarrow \text{id} \tilde{\otimes} \Phi_2^{abc} \\
(\Psi_1^{ab}(x) \tilde{\otimes} \Psi_1^{bc}(y)) \tilde{\otimes} \alpha_1^c & \xrightarrow[\Psi_2^{abc} \tilde{\otimes} \text{id}]{} \Psi_1^{ac}(x \otimes y) \tilde{\otimes} \alpha_1^c \xrightarrow[\alpha_2^{ac}]{} \alpha_1^a \tilde{\otimes} \Phi_1^{ac}(x \otimes y)
\end{array} \tag{2.12a}$$

and

$$\begin{array}{ccc}
\text{id}_{\Psi_0(a)} \otimes \alpha_1^a & \xrightarrow{r} & \alpha_1^a \xrightarrow{l^{-1}} \alpha_1^a \otimes \text{id}_{\Phi_0(a)} \\
\downarrow \Psi_2^a \otimes \text{id} & & \downarrow \text{id} \otimes \Phi_2^a \\
\Psi_1^{aa}(\text{id}_a) \otimes \alpha_1^a & \xrightarrow[\alpha_2^{aa}]{} & \alpha_1^a \otimes \Phi_1^{aa}(\text{id}_a)
\end{array} \tag{2.12b}$$

are commutative.

Definition 2.13. A lax natural 2-transformation $\alpha = (\alpha_1, \alpha_2)$ for which the 2-cells α_2 are natural isomorphisms is called a weak natural 2-transformation.⁴ A lax natural 2-transformation $\alpha = (\alpha_1, \alpha_2)$ for which the 2-cells α_2 are trivial is called a strict natural 2-transformation.

The composition of natural 2-transformations is governed by the following proposition.

Proposition 2.14. Given three lax 2-functors $\Phi, \Psi, \Xi : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$ between two weak 2-categories \mathcal{B} and $\tilde{\mathcal{B}}$ and two lax natural 2-transformations $\alpha : \Phi \Rightarrow \Psi$ and $\beta : \Psi \Rightarrow \Xi$, then there is a lax natural 2-transformation $\gamma : \Phi \Rightarrow \Xi$ such that

$$\begin{array}{ccc}
\begin{array}{ccc}
\Phi_0(b) & \xrightarrow{\Phi_1^{ab}(x)} & \Phi_0(a) \\
\alpha_1^b \downarrow & \nearrow \alpha_2^{ab}(x) & \downarrow \alpha_1^a \\
\Psi_0(b) & \xrightarrow{\Psi_1^{ab}(x)} & \Psi_0(a) \\
\beta_1^b \downarrow & \nearrow \beta_2^{ab}(x) & \downarrow \beta_1^a \\
\Xi_0(b) & \xrightarrow{\Xi_1^{ab}(x)} & \Xi_0(a)
\end{array} & = & \begin{array}{ccc}
\Phi_0(b) & \xrightarrow{\Phi_1^{ab}(x)} & \Phi_0(a) \\
\gamma_1^b \downarrow & \nearrow \gamma_2^{ab}(x) & \downarrow \gamma_1^a \\
\Xi_0(b) & \xrightarrow{\Xi_1^{ab}(x)} & \Xi_0(a)
\end{array}
\end{array} \tag{2.13a}$$

with $\gamma_1^a : \Phi_0(a) \rightarrow \Xi_0(a)$ and $\gamma_2^{ab} : \Xi_1^{ab}(x) \tilde{\otimes} \gamma_1^b \Rightarrow \gamma_1^a \tilde{\otimes} \Phi_1^{ab}(x)$ and

$$\begin{aligned}
\gamma_1^a &= \beta_1^a \tilde{\otimes} \alpha_1^a, \\
\gamma_2^{ab} &= \tilde{a}_{\beta_1^a, \alpha_1^a, \Phi^{ab}(x)}^{-1} \tilde{\otimes} (\text{id}_{\beta_1^a} \tilde{\otimes} \alpha_2^{ab}(x)) \tilde{\otimes} \tilde{a}_{\beta_1^a, \Psi^{ab}(x), \alpha_1^b} \tilde{\otimes} (\beta_2^{ab}(x) \tilde{\otimes} \text{id}_{\alpha_1^b}) \tilde{\otimes} \tilde{a}_{\Xi^{ab}(x), \beta_1^b, \alpha_1^b}^{-1}
\end{aligned} \tag{2.13b}$$

⁴Weak natural 2-transformations are also known as pseudo-natural transformations.

for all $a, b \in B_0$ and $x \in C_0(a, b)$.

Proof: It is straightforward to see that $\gamma = (\gamma_1, \gamma_2)$ given in (2.13b) is a map $\gamma_1^a : \Phi_0(a) \rightarrow \Xi_0(a)$ and $\gamma_2^{ab} : \Xi_1^{ab}(x) \tilde{\otimes} \gamma_1^b \Rightarrow \gamma_1^a \tilde{\otimes} \Phi_1^{ab}(x)$ between the lax 2-functors Φ and Ξ . That this is indeed a lax natural 2-transformation is a consequence of the pasting theorem for weak 2-categories, see Verity [27]. \square

Finally, for 2-categories, it is useful to continue the sequence of 2-categories, 2-functors, 2-transformations to 2-modifications.

Definition 2.15. Let $\Phi, \Psi : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$ be two lax 2-functors between two weak 2-categories \mathcal{B} and $\tilde{\mathcal{B}}$. A 2-modification between two lax natural 2-transformations $\alpha, \beta : \Phi \rightarrow \Psi$ is a collection of morphisms $\varphi_a : \alpha_a \Rightarrow \beta_a$ such that

$$\begin{array}{ccc} \Psi_1^{ab}(x) \tilde{\otimes} \alpha_1^b & \xrightarrow{\text{id} \tilde{\otimes} \varphi_b} & \Psi_1^{ab}(x) \tilde{\otimes} \beta_1^b \\ \alpha_2^{ab} \Downarrow & & \Downarrow \beta_2^{ab} \\ \alpha_1^a \tilde{\otimes} \Phi_1^{ab}(x) & \xrightarrow{\varphi_a \tilde{\otimes} \text{id}} & \beta_1^a \tilde{\otimes} \Phi_1^{ab}(x) \end{array} \quad (2.14)$$

is commutative. If the morphisms φ_a are invertible, we call the 2-modification invertible.

Note that composition of 2-modifications is trivially obtained by concatenation.

2.2. Weak 2-groupoids

In this section, we would like to introduce the notion of 2-groupoids as they are the key to defining principal 2-bundles. To this end, we wish to recall the definition of a groupoid first.

Definition 2.16. A groupoid is a small category in which every morphism is invertible.

Two important examples of groupoids that we shall frequently encounter throughout this work are those of the Čech groupoid and the delooping of a group.

Example 2.17. The Čech groupoid relative to a covering $\mathfrak{U} = \{U_a\}$ of a topological manifold X , denoted by $\check{\mathcal{C}}(\mathfrak{U})$ in the following, is defined to be the groupoid that has the covering sets as objects and the intersection of covering sets as morphisms. Concretely, the set of objects of $\check{\mathcal{C}}(\mathfrak{U})$ is defined to be the disjoint union $\dot{\bigcup}_a U_a := \bigcup_a \{(x, a) \mid x \in U_a\}$ and the set of morphisms of $\check{\mathcal{C}}(\mathfrak{U})$ is defined to be the disjoint union $\dot{\bigcup}_{a,b} U_a \cap U_b := \bigcup_{a,b} \{(x, a, b) \mid x \in U_a \cap U_b\}$, together with the structure maps

$$\begin{aligned} s(x, a, b) &:= (x, b), \quad t(x, a, b) := (x, a), \quad \text{id}_{(x,a)} := (x, a, a), \\ (x, a, b) \circ (x, b, c) &:= (x, a, c). \end{aligned} \quad (2.15)$$

Example 2.18. Let G be a group. The delooping of G , denoted by BG , is defined to be the groupoid that has only a single object, denoted by e , and the group G as its morphisms, $g : e \rightarrow e$ with $g \in G$. In BG , the composition of morphisms is then simply given by the group multiplication on G , that is, $g_2 \circ g_1 := g_2 g_1$ for any $g_{1,2} \in G$.

We shall be interested in the categorification of the notion of a groupoid, which is defined as follows.

Definition 2.19. A weak 2-groupoid is a weak 2-category such that all morphisms are equivalences. A weak 2-groupoid with an underlying strict 2-category is called a strict 2-groupoid.

All morphisms being equivalences implies that the 2-cells are strictly invertible and the 1-cells are invertible up to isomorphisms. Unpacking this definition further⁵, a weak 2-groupoid is a weak 2-category \mathcal{B} such that for every pair of objects $a, b \in B_0$, the category $\mathcal{C}(a, b)$ is a groupoid. Moreover, for every pair $a, b \in B_0$ there is a functor $\bar{\cdot} : \mathcal{C}(a, b) \rightarrow \mathcal{C}(b, a)$ and for every 1-cell $x \in C_0(a, b)$, there are natural isomorphisms $i_x : \text{id}_a \Rightarrow x \otimes \bar{x}$ and $e_x : \bar{x} \otimes x \Rightarrow \text{id}_b$ called the unit and counit. These have to satisfy coherence axioms, which state that for any 1-cell $x \in C_0(a, b)$ and $a, b \in B_0$, the diagrams

$$\begin{array}{ccc} (x \otimes \bar{x}) \otimes x & \xrightarrow{\quad a \quad} & x \otimes (\bar{x} \otimes x) \\ i^{-1} \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes e \\ \text{id}_a \otimes x & \xrightarrow{\quad l \quad} x \xleftarrow{\quad r \quad} x \otimes \text{id}_b & \end{array} \quad (2.16a)$$

and

$$\begin{array}{ccc} (\bar{x} \otimes x) \otimes \bar{x} & \xrightarrow{\quad a \quad} & \bar{x} \otimes (x \otimes \bar{x}) \\ e \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes i^{-1} \\ \text{id}_b \otimes \bar{x} & \xrightarrow{\quad r \quad} x \xleftarrow{\quad l \quad} \bar{x} \otimes \text{id}_a & \end{array} \quad (2.16b)$$

are commutative.

Example 2.20. An example of a strict Lie 2-groupoid important in our subsequent discussion is the so-called Čech 2-groupoid. The 0- and 1-cells are given by the objects and morphisms of the Čech groupoid (see Example 2.17), and all 2-cells defined to be trivial.

In Example 2.18, we have seen that any group can be viewed as a groupoid with a single object. Analogously, we give the following definition.

Definition 2.21. A weak 2-group is a weak 2-groupoid with a single 0-cell.

⁵cf. Hardie et al. [28]

Remark 2.22. *This definition is equivalent to the definition given by Baez & Lauda [4]. In particular, they define weak 2-groups as weak monoidal categories in which all morphisms are invertible and all objects are weakly invertible. They also introduce so-called coherent 2-groups as weak monoidal categories in which all morphisms are invertible and all objects come with an adjoint equivalence. Both notions are shown to be equivalent. Our definition 2.21 is actually the delooping of a coherent Lie 2-group in the sense of Baez & Lauda, cf. Example 2.18, and we shall therefore write $\mathbf{BG} = (\{e\}, M, N)$. The single 0-cell is denoted by e in the following while the 1- and 2-cells are denoted by M and N , respectively.*

Definition 2.23. *A strict 2-group is a strict 2-groupoid with a single 0-cell.*

Put differently, a strict 2-group is a weak 2-group in which all unitors, units, counits and associators are trivial. Furthermore, we will need the notion of a skeletal 2-group which is as follows.

Definition 2.24. *A skeletal 2-group is a weak 2-group, in which the contained category $\mathcal{C}(e, e)$ is skeletal.*

Recall that a category is skeletal, if all isomorphic objects are equal: for all morphisms f in the category, $s(f) = t(f)$.

One version of Mac Lane's coherence theorem [29] states that every weak monoidal category is equivalent to a strict monoidal category. In the case of weak 2-groups, we have the following proposition from [4, Sec. 8.3], which can be used to classify weak Lie 2-groups.

Proposition 2.25. *(Baez & Lauda [4]) Every weak 2-group is categorically equivalent to a 'special' weak 2-group which is skeletal and in which all unitors, units, and counits are identity natural transformations. In particular, a special weak 2-group can be given in terms of a group G , an Abelian group H , a representation α of G as automorphisms of H and an element $[a] \in H^3(G, H)$.*

In addition, we have the following result.

Proposition 2.26. *(Baez & Lauda [4]) Every weak 2-group is categorically equivalent to a strict 2-group.*

The notion of 2-groups relevant for our subsequent discussion will be the following.

Definition 2.27. *A semistrict 2-group is a weak 2-group in which the unitors, units, and counits are all trivial.*

We would like to emphasize that this notion is weaker than that of a strict 2-group, because the associators remain unrestricted. For semistrict 2-groups, we have the following results.

Proposition 2.28. *In any semistrict 2-group $\mathcal{BG} = (\{e\}, M, N)$, the associators $\mathbf{a}_{\text{id}_e, m, m'}$, $\mathbf{a}_{m, m', \text{id}_e}$, $\mathbf{a}_{m, \text{id}_e, m'}$, $\mathbf{a}_{m, \overline{m}, m}$, and $\mathbf{a}_{\overline{m}, m, \overline{m}}$ are all trivial for all $m, m' \in M$.*

Proof: This follows trivially by combining the pentagon and triangle diagrams with the diagrams displayed in (2.16). \square

Proposition 2.29. *In any semistrict 2-group $\mathcal{BG} = (\{e\}, M, N)$ and for any 2-cell $n \in N$,*

$$n^{-1} = \mathbf{a}_{\mathbf{s}(n), \overline{\mathbf{t}(n)}, \mathbf{t}(n)} \circ ((\text{id}_{\mathbf{s}(n)} \otimes \bar{n}) \otimes \text{id}_{\mathbf{t}(n)}) : \mathbf{t}(n) \Rightarrow \mathbf{s}(n) \quad (2.17)$$

such that $n \circ n^{-1} = \text{id}_{\mathbf{t}(n)}$ and $n^{-1} \circ n = \text{id}_{\mathbf{s}(n)}$.

Proof: This follows from the proof of Proposition 20 in [4]. \square

2.3. Lie 2-groups

To restrict the rather general notion of a groupoid, we can regard Lie groupoids as groupoids internal to a certain category \mathcal{K} . In general, a category internal to $\mathcal{K} = (K_0, K_1)$ consists of an object of objects and an object of morphisms, which are both elements in K_0 . The structure maps $\mathbf{s}, \mathbf{t}, \text{id}, \circ$ are given in terms of elements of K_1 and all commutative diagrams which hold in a category also hold in the internalised category. Internal functors and modifications are defined in an analogous manner. A groupoid internal to a category \mathcal{K} is simply a category internal to \mathcal{K} , in which all the morphisms are strictly invertible.

In this manner, we can define, for instance, topological groupoids as groupoids in **Top**, the category of topological spaces and continuous functions between them. Similarly, Lie groupoids are defined as follows.

Definition 2.30. *A Lie groupoid is a groupoid in **Diff**, the category of smooth manifolds and smooth functions between them.*

Thus, Lie groupoids are groupoids in which the sets of objects and morphisms are smooth manifolds and all the structure maps are smooth.

Remark 2.31. *Recall that for any category \mathcal{K} there exists a strict 2-category $\mathcal{K}\mathbf{Cat}$ with objects being categories internal to \mathcal{K} , morphisms being functors in \mathcal{K} and 2-morphisms being natural transformations in \mathcal{K} . In particular, **DiffCat** is the strict 2-category with Lie groupoids as 0-cells, strict Lie groupoid morphisms as 1-cells and 2-morphisms of Lie groupoids as 2-cells.*

We can now define Lie 2-groups by internalising weak 2-groups.

Definition 2.32. A Lie 2-group is a weak 2-group in DiffCat .

That is, a Lie 2-group consists of an object C in DiffCat , a multiplication morphism $\otimes : C \times C \rightarrow C$, an identity object $\mathbb{1}$ and an inverse map $\bar{\cdot} : C \rightarrow C$. Furthermore, we have for all objects x, y, z in the category C the following natural isomorphisms: an associator $\mathbf{a} : (x \otimes y) \otimes z \Rightarrow x \otimes y \otimes z$, left and right unitors $\mathbf{l}_x : \mathbb{1} \otimes x \rightarrow x$ and $\mathbf{r}_x : x \otimes \mathbb{1} \Rightarrow x$ as well as a unit and counit $\mathbf{u}_x : \mathbb{1} \Rightarrow x \otimes \bar{x}$ and $\mathbf{u}'_x : \bar{x} \otimes x \rightarrow \mathbb{1}$, such that the pentagon and triangle identities as well as the first and second zig-zag identities are satisfied, cf. [4].

For our purposes, we wish to restrict the notion of a Lie 2-group as given Definition 2.32 somewhat further.

Definition 2.33. A semistrict Lie 2-group is a coherent 2-group \mathcal{G} in DiffCat with trivial unitors, units and counits.

Note that by Proposition 2.25, semistrict Lie 2-groups are still categorically equivalent to weak Lie 2-groups.

Definition 2.34. A strict Lie 2-group is a weak Lie 2-group with underlying strict Lie 2-groupoid.

Concretely, in a strict Lie 2-group, the associators as well as the left and right unitors are all identity morphisms. There is an equivalent formulation of strict Lie 2-groups in terms of crossed modules of Lie groups.

Definition 2.35. A crossed module of Lie groups is a pair of Lie groups (\mathbf{H}, \mathbf{G}) together with a Lie group homomorphism⁶ $\partial : \mathbf{H} \rightarrow \mathbf{G}$ and an action \triangleright of \mathbf{G} on \mathbf{H} by automorphisms. The map ∂ is \mathbf{G} -equivariant and satisfies the Peiffer identity. Explicitly, we have

$$\partial(g \triangleright h) = g\partial(h)g^{-1} \quad \text{and} \quad \partial(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1} \quad (2.18)$$

for all $g \in \mathbf{G}$ and $h, h_1, h_2 \in \mathbf{H}$.

Then we have the following result.

Proposition 2.36. A strict Lie 2-group is equivalent to a crossed module of Lie groups.

See Baez & Lauda [4] for a detailed proof. We shall use an identification between strict Lie 2-groups and crossed modules of Lie groups that slightly differs from that of [4]. Given a crossed module of Lie groups $(\mathbf{H} \xrightarrow{\partial} \mathbf{G}, \triangleright)$, we obtain a strict Lie 2-group $\mathbf{B}\mathcal{G} = (\{e\}, M, N)$,

⁶This homomorphism is often denoted by \mathbf{t} . Here, however, to avoid confusion with the source and target maps of the weak 2-group, we use the symbol ∂ .

by identifying $M := G$ and $N := H \rtimes G$ and setting $s(h, g) := \partial(h^{-1})g$ and $t(h, g) := g$ for $h \in H$ and $g \in G$ together with

$$\begin{aligned} g_2 \otimes g_1 &:= g_2 g_1 , \\ (h_2, g_2) \otimes (h_1, g_1) &:= ((g_2 \triangleright h_1)h_2, g_2 g_1) , \\ (h_2, g) \circ (h_1, \partial(h_2^{-1})g) &:= (h_2 h_1, g) . \end{aligned} \tag{2.19}$$

On the other hand, given a strict Lie 2-group $B\mathcal{G} = (\{e\}, M, N)$, we define a crossed module $(H \xrightarrow{\partial} G, \triangleright)$ by putting $H := \ker(t)$ and $G := M$ and

$$\begin{aligned} \partial &:= s|_{\ker(t)} , \\ g \triangleright h &:= \text{id}_g \otimes h \otimes \text{id}_{g^{-1}} . \end{aligned} \tag{2.20}$$

2.4. Lie 2-algebras

Apart from Lie 2-groups, we shall also be dealing with Lie 2-algebras. The most general kind of Lie 2-algebra currently in use has been defined by Roytenberg [30] as follows.

Definition 2.37. A weak Lie 2-algebra is a linear category $\mathcal{L} = (L_0, L_1)$ together with

- (i) the bracket, a bilinear functor $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$,
- (ii) the alternator, a bilinear natural transformation $S : [X, Y] \Rightarrow -[Y, X]$ and
- (iii) the Jacobiator, a trilinear natural transformation $J : [X, [Y, Z]] \Rightarrow [[X, Y], Z] + [Y, [X, Z]]$ for $X, Y, Z \in L_0$.

These structure maps have to satisfy a number of coherence axioms, cf. [30].

In this paper, we are merely interested in so-called semistrict Lie 2-algebras.

Definition 2.38. A semistrict Lie 2-algebra is a weak Lie 2-algebra in which the alternator is trivial.

Instead of working directly with semistrict Lie 2-algebras and their rather involved coherence axioms, we can switch to a categorically equivalent formulation in terms of 2-term L_∞ -algebras, as was shown in [31]. The general definition of a strong homotopy Lie algebra is given in appendix A. Here, we just recall the following definition

Definition 2.39. A 2-term L_∞ -algebra consists of a two-term complex of vector spaces V and W ,

$$V \xrightarrow{\mu_1} W \xrightarrow{\mu_2} 0 , \tag{2.21}$$

where we associate gradings -1 and 0 to elements of V and W , respectively. This complex is equipped with higher products μ_1, μ_2, μ_3 , which vanish except for

$$\begin{aligned}\mu_1 : V &\rightarrow W, & \mu_2 : W \wedge W &\rightarrow W, & \mu_2 : V \wedge W &\rightarrow V, \\ \mu_3 : W \wedge W \wedge W &\rightarrow V.\end{aligned}\tag{2.22}$$

Moreover, these products are required to satisfy the following homotopy Jacobi identities:

$$\begin{aligned}\mu_1(\mu_2(w, v)) &= \mu_2(w, \mu_1(v)), \\ \mu_2(\mu_1(v_1), v_2) &= \mu_2(v_1, \mu_1(v_2)), \\ \mu_1(\mu_3(w_1, w_2, w_3)) &= -\mu_2(\mu_2(w_1, w_2), w_3) - \mu_2(\mu_2(w_3, w_1), w_2) - \mu_2(\mu_2(w_2, w_3), w_1), \\ \mu_3(\mu_1(v), w_1, w_2) &= -\mu_2(\mu_2(w_1, w_2), v) - \mu_2(\mu_2(v, w_1), w_2) - \mu_2(\mu_2(w_2, v), w_1), \\ \mu_2(\mu_3(w_1, w_2, w_3), w_4) - \mu_2(\mu_3(w_4, w_1, w_2), w_3) + \mu_2(\mu_3(w_3, w_4, w_1), w_2) - \\ &\quad - \mu_2(\mu_3(w_2, w_3, w_4), w_1) = \\ &= \mu_3(\mu_2(w_1, w_2), w_3, w_4) - \mu_3(\mu_2(w_2, w_3), w_4, w_1) + \mu_3(\mu_2(w_3, w_4), w_1, w_2) - \\ &\quad - \mu_3(\mu_2(w_4, w_1), w_2, w_3) - \mu_3(\mu_2(w_1, w_3), w_2, w_4) - \mu_3(\mu_2(w_2, w_4), w_1, w_3),\end{aligned}\tag{2.23}$$

where $v, v_i \in V$ and $w, w_i \in W$.

Remark 2.40. Note that for every 2-term L_∞ -algebra $L = (V, W, \mu_1, \mu_2, \mu_3)$, there is another 2-term L_∞ -algebra \tilde{L} with the same underlying vector spaces $\tilde{V} = V$, $\tilde{W} = W$, but with higher products $\tilde{\mu}_1 = -\mu_1$, $\tilde{\mu}_2 = \mu_2$ and $\tilde{\mu}_3 = -\mu_3$.

Example 2.41. A typical example of a semistrict Lie 2-algebra is the string algebra of a Lie algebra \mathfrak{g} . Here, $W = \mathfrak{g}$, $V = \mathbb{R}$ and the only non-trivial higher products are $\mu_2(w_1, w_2) = [w_1, w_2]$ and $\mu_3(w_1, w_2, w_3) = \langle w_1, [w_2, w_3] \rangle$, where $w_1, w_2, w_3 \in W$ and $\langle \cdot, \cdot \rangle$ is the Killing form on \mathfrak{g} .

Let us briefly recall the details of the equivalence between semistrict Lie 2-algebras and 2-term L_∞ -algebras.⁷ We start from a Lie 2-algebra $\mathcal{L} = (L_0, L_1)$ and put

$$V := \ker(t) \subseteq L_1, \quad W := L_0, \quad \text{and} \quad \mu_1 := s|_V.\tag{2.24}$$

The higher products are defined as follows:

$$\begin{aligned}\mu_2(w_1, w_2) &:= [w_1, w_2], & \mu_2(w, v) &= -\mu_2(v, w) := [\text{id}_w, v], \\ \mu_3(w_1, w_2, w_3) &:= J(w_1, w_2, w_3) - \text{id}_{[[w_1, w_2], w_3] + [w_2, [w_1, w_3]]},\end{aligned}\tag{2.25}$$

⁷A similar equivalence exists for weak Lie 2-algebras [30], but the resulting normalised chain complex is less convenient to work with.

where $w_1, w_2, w_3, w \in W$ and $v \in V$. This map from a semistrict Lie 2-algebra to a 2-term L_∞ -algebra can be extended to a functor Φ between the corresponding categories.

Inversely, given a 2-term L_∞ -algebra $V \xrightarrow{\mu_1} W$, we obtain a semistrict Lie 2-algebra $\mathcal{L} = (L_0, L_1)$ by putting

$$\begin{aligned} L_0 &:= W, \quad L_1 := V \oplus W, \quad \mathfrak{s}(v, w) := w - \mu_1(v), \quad \mathfrak{t}(v, w) := w, \\ \text{id}_w &:= (0, w), \quad (v_2, w) \circ (v_1, w - \mu_1(v_2)) := (v_1 + v_2, w) \end{aligned} \quad (2.26)$$

for all $v, v_1, v_2 \in V$ and $w \in W$. In addition, we set

$$\begin{aligned} [w_1, w_2] &:= \mu_2(w_1, w_2), \\ [(v_1, w_1), (v_2, w_2)] &:= (\mu_2(v_1, w_2) + \mu_2(w_1 - \mu_1(v_1), v_2), \mu_2(w_1, w_2)), \\ J(w_1, w_2, w_3) &:= (\mu_3(w_1, w_2, w_3), -\mu_2(\mu_2(w_1, w_2), w_3) - \mu_2(\mu_2(w_3, w_1), w_2)). \end{aligned} \quad (2.27)$$

Again, this map from a 2-term L_∞ -algebra to a semistrict Lie 2-algebra can be extended to a functor Ψ between the corresponding categories.

We have the following results.

Proposition 2.42. *(Baez & Crans [31]) Together, the functors Φ and Ψ defined above can be shown to form an equivalence, which can even be extended to an equivalence of 2-categories.*

Proposition 2.43. *(Baez & Crans [31]) There is a one-to-one correspondence between equivalence classes of semistrict Lie 2-algebras and ‘special’ 2-term L_∞ -algebras given in terms of a Lie algebra \mathfrak{g} , a representation of \mathfrak{g} on a vector space V and an element J of $H^3(\mathfrak{g}, V)$. Here, $\mu_1 = 0$, μ_2 is the Lie bracket in \mathfrak{g} or the action on V and $\mu_3 = J$.*

Semistrict Lie 2-algebras can be restricted further to obtain strict Lie 2-algebras.

Definition 2.44. *A strict Lie 2-algebra is a weak Lie 2-algebra with trivial alternator and trivial Jacobiator.*

Our above discussion immediately implies that strict Lie 2-algebras are equivalent to 2-term L_∞ -algebras with trivial product μ_3 , which in turn, can be encoded in a differential crossed module.

Definition 2.45. *The differential crossed module of a crossed module of Lie groups is obtained by applying the tangent functor to the crossed module.*

In particular, given a crossed module of Lie groups $(H \xrightarrow{\partial} G, \triangleright)$, the tangent functor yields a differential crossed module⁸ $(\mathfrak{h} \xrightarrow{\partial} \mathfrak{g}, \triangleright)$, where $\mathfrak{h} := \text{Lie}(H)$ and $\mathfrak{g} := \text{Lie}(G)$. The maps ∂

⁸Our notation does not distinguish between the maps ∂, \triangleright and their differentials.

and \triangleright satisfy

$$\partial(X \triangleright Y) = [X, \partial(Y)] \quad \text{and} \quad \partial(Y_1) \triangleright Y_2 = [Y_1, Y_2] , \quad (2.28)$$

where $X \in \mathfrak{g}$ and $Y, Y_{1,2} \in \mathfrak{h}$.

The differential crossed module corresponding to a 2-term L_∞ -algebra $V \xrightarrow{\mu_1} W$ with trivial μ_3 is obtained by identifying $\mathfrak{h}, \mathfrak{g}$ and ∂ with V, W and μ_1 as well as

$$[w_1, w_2] := \mu_2(w_1, w_2) , \quad v \triangleright w := \mu_2(v, w) \quad \text{and} \quad [v_1, v_2] := \mu_2(\mu_1(v_1), v_2) \quad (2.29)$$

for $v_1, v_2, v \in V = \mathfrak{h}$ and $w_1, w_2, w \in W = \mathfrak{g}$. This identification is readily inverted.

3. Principal 2-bundles with Lie 2-groups

We come now to the discussion of principal 2-bundles with weak structure 2-groups over smooth manifolds. An earlier description of general 2-bundles from a slightly different point of view can be found in Bartels [3].

For the following discussion, let X be a smooth manifold and let $\mathfrak{U} = \{U_a\}$ be a covering of X .

3.1. Principal bundles as functors

Recall that a Čech p -cochain with values in a group \mathbf{G} on a manifold X with respect to the cover \mathfrak{U} is a set of functions on all non-empty overlaps $U_{a_1} \cap \cdots \cap U_{a_{p+1}}$ with values in \mathbf{G} . We then give the following definition.

Definition 3.1. *A (non-Abelian) Čech 1-cocycle is a Čech 1-cochain $\{g_{ab}\}$ consisting of maps⁹ $g_{ab} : U_a \cap U_b \rightarrow \mathbf{G}$ such that*

$$g_{ab}g_{bc} = g_{ac} \quad \text{on} \quad U_a \cap U_b \cap U_c . \quad (3.1)$$

Two Čech 1-cocycles $\{g_{ab}\}$ and $\{\tilde{g}_{ab}\}$ are cohomologous or equivalent if and only if there is a Čech 0-cochain $\{g_a\}$ consisting of maps $g_a : U_a \rightarrow \mathbf{G}$ such that

$$g_{ab} = g_a \tilde{g}_{ab} g_b^{-1} . \quad (3.2)$$

The first Čech cohomology set, denoted by $H^1(\mathfrak{U}, \mathbf{G})$, is defined as the set of Čech 1-cocycles modulo this equivalence.

⁹If not stated otherwise, we shall always assume that intersections of patches are non-empty.

Čech cohomology sets can be rendered independent of the cover by taking direct limit over all open covers \mathfrak{U} of X . We then write $H^1(X, \mathbf{G})$ instead of $H^1(\mathfrak{U}, \mathbf{G})$, that is,

$$H^1(X, \mathbf{G}) = \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, \mathbf{G}) . \quad (3.3)$$

Elements of $H^1(X, \mathbf{G})$ are also known as (sets of) transition functions of principal bundles with structure group \mathbf{G} (or principal \mathbf{G} -bundles for short), and we can identify a principal \mathbf{G} -bundle over X with an element in $H^1(X, \mathbf{G})$. To allow for a categorification of this picture, we switch to a functorial description of principal fibre bundles.

Definition 3.2. *A principal bundle with structure group \mathbf{G} is a smooth functor from the Čech groupoid to the Lie groupoid \mathbf{BG} .¹⁰ Any two principal bundles are called equivalent if and only if there is a natural isomorphism between the corresponding functors.*

Again, we can remove the dependence on the cover in this definition by taking direct limits.

Definition 3.2 is well-known from the description of principal bundles in terms of classifying spaces [32]. Explicitly, we have a functor

$$\Phi : \mathcal{C}(\mathfrak{U}) \rightarrow \mathbf{BG} \quad (3.4)$$

and we set $e_a := \Phi(x, a)$ and $g_{ab} := \Phi(x, a, b)$. Because Φ is a functor, we immediately arrive at the cocycle conditions (3.1) as well as $\Phi(x, a, a) = \text{id}_{\Phi(x, a)} = \mathbb{1}_{\mathbf{G}} \in \mathbf{G}$. In addition, two functors Φ and Ψ corresponding to principal bundles are equivalent if and only if there is a natural isomorphism $\alpha : \tilde{\Phi} \rightarrow \Phi$. Defining $e_a := \Phi(x, a)$, $g_{ab} = \Phi(x, a, b)$ and $\tilde{g}_a := \alpha_{(x, a)} : \tilde{\Phi}(x, a) \rightarrow \Phi(x, a)$, the following diagram is commutative:

$$\begin{array}{ccc} \tilde{e}_b & \xrightarrow{\tilde{g}_{ab}} & \tilde{e}_a \\ g_b \downarrow & & \downarrow g_a \\ e_b & \xrightarrow{g_{ab}} & e_a \end{array} \quad (3.5)$$

In formulæ, this is

$$g_a \tilde{g}_{ab} = g_{ab} g_b , \quad (3.6)$$

which amounts to (3.2). We thus arrive at the following statement, which motivates our Definition 3.2.

Proposition 3.3. *Denoting the set of equivalence classes of smooth functors between $\mathcal{C}(\mathfrak{U})$ and \mathbf{BG} by $[\mathcal{C}(\mathfrak{U}) \rightarrow \mathbf{BG}]$, we have*

$$H^1(\mathfrak{U}, \mathbf{G}) \cong [\mathcal{C}(\mathfrak{U}) \rightarrow \mathbf{BG}] . \quad (3.7)$$

¹⁰See Examples 2.17 and 2.18 for the relevant definitions.

Other conventional definitions are now also straightforwardly rephrased.

Definition 3.4. A principal bundle is called trivial if and only if the underlying functor is equivalent to the functor

$$e_a : (x, a) \mapsto e \quad \text{and} \quad g_{ab} : (x, a, b) \mapsto \mathbb{1}_G . \quad (3.8)$$

Concretely, a principal bundle is trivial if and only if there is a natural isomorphism $\alpha = \{g_a\}$ such that

$$g_a = g_{ab}g_b . \quad (3.9)$$

Finally, let $\phi : X \rightarrow Y$ be a smooth map between two smooth manifolds X and Y . Let \mathfrak{U}_Y be a covering of Y . Then we can construct a covering \mathfrak{U}_X of X from the pre-images of the patches in \mathfrak{U}_Y under ϕ . This yields a morphism of groupoids $\check{\mathcal{C}}(\mathfrak{U}_X) \rightarrow \check{\mathcal{C}}(\mathfrak{U}_Y)$.

Definition 3.5. The pullback of a principal bundle Φ over Y with respect to an open cover \mathfrak{U}_Y along a smooth map $\phi : X \rightarrow Y$ is the composition $\Phi \circ \phi_{\mathfrak{U}}$, where $\phi_{\mathfrak{U}}$ is the groupoid morphism induced by ϕ .

Definition 3.6. The restriction of a principal bundle Φ over a manifold X to a submanifold Y of X is the pullback of Φ along the embedding map $Y \hookrightarrow X$.

3.2. Principal 2-bundles as 2-functors

The reformulation of principal bundles with structure group G in terms of functors between the Čech groupoid and the Lie groupoid BG is a good starting point for categorifying the notion of principal bundles. We can simply regard the Čech groupoid as an n -groupoid and take an n -functor to a Lie n -groupoid with a single 0-cell. In the following, we shall develop the case $n = 2$ in detail. Note that our discussion will first centre around weak principal 2-bundles which we shall define in terms of weak 2-functors. Consider a weak Lie 2-group $B\mathcal{G} = (\{e\}, M, N)$ which is a weak Lie 2-groupoid with a single object e . As in Section 2, we shall denote horizontal and vertical composition in $B\mathcal{G}$ by \otimes and \circ , respectively.

Principal 2-bundles will be described by Čech cocycles with values in $B\mathcal{G}$. We therefore start by giving the following definition.

Definition 3.7. A degree- p Čech cochain with values in a weak Lie 2-group $B\mathcal{G} = (\{e\}, M, N)$ consists of a degree- $(p-1)$ Čech cochain $\{m_{a_0 \dots a_{p-1}}\}$ with values in M , a degree- p Čech cochain $\{n_{a_0 \dots a_p}\}$ with values in N and a degree- $(p-2)$ Čech cochain $\{n_{a_0 \dots a_{p-2}}\}$ with values in N .

In the following, we shall be interested in the case $p = 2$, for which we have a triple

$$(\{m_{ab}\}, \{n_{abc}\}, \{n_a\}) . \quad (3.10)$$

These cochains generalise the usual Čech cochains appearing in the definition of an ordinary principal bundle in the way that is familiar from strict principal 2-bundles: the $\{m_{ab}\}$ are generalised transition functions on overlaps, the $\{n_{abc}\}$ are the gluing isomorphisms on triple overlaps and the $\{n_a\}$ are the isomorphisms between the unit in M and the trivial transition functions $\{m_{aa}\}$.

To derive the explicit cocycle and coboundary conditions appropriate for weak Lie 2-groups, we shall again employ the functorial approach.

Definition 3.8. *A weak principal 2-bundle with structure 2-group \mathcal{BG} with respect to the cover \mathfrak{U} of a manifold X is a weak 2-functor between the Čech 2-groupoid $\check{\mathcal{C}}(\mathfrak{U})$ and \mathcal{BG} .*

Let us be more specific. We have a weak 2-functor¹¹

$$\Phi : \check{\mathcal{C}}(\mathfrak{U}) \rightarrow \mathcal{BG} \quad (3.11)$$

consisting of a function $\Phi_0(x, a)$, functors $\Phi_1(x, a, b)$ and 2-cells Φ_2 . Note that the 0-cells of $\mathcal{BG} = (\{e\}, M, N)$ and the 2-cells of $\check{\mathcal{C}}(\mathfrak{U})$ are trivial and we shall denote them by e . We can therefore specify Φ in terms of constant functions $e_a := \Phi_0(x, a) : U_a \rightarrow e$, functions $m_{ab} := \Phi_1(x, a, b)|_M : U_a \cap U_b \rightarrow M$, constant functions $e_{ab} := \Phi_1(x, a, b)|_N : e \rightarrow \text{id}_{\text{id}_e}$ and invertible functions $n_{abc} : U_a \cap U_b \cap U_c \rightarrow N$ and $n_a : U_a \rightarrow N$ describing the 2-cell Φ_2 . Because $\text{id}_{(x,a)} = (x, a, a)$, we have by definition $\Phi_1(\text{id}_{(x,a)}) = \Phi_1(x, a, a) = m_{aa}$. The fact that Φ_1 is a functor implies $\text{id}_{m_{ab}} = \text{id}_{\Phi_1(x,a,b)} = \Phi_1(\text{id}_{(x,a,b)})$. Finally, with $\Phi_1((x, a, b) \circ (x, b, c)) = \Phi_1(x, a, c) = m_{ac}$, we have the natural isomorphisms

$$\begin{aligned} n_{abc} : m_{ab} \otimes m_{bc} &\Rightarrow m_{ac} , \\ n_a : \text{id}_{e_a} &\Rightarrow m_{aa} , \end{aligned} \quad (3.12)$$

with $\text{id}_{e_a} \in M$.

The following diagrams, which arise from (2.9) with \mathbf{a} , \mathbf{r} , and \mathbf{l} being trivial in $\check{\mathcal{C}}(\mathfrak{U})$, are commutative:

$$\begin{array}{ccc} m_{ab} \otimes (m_{bc} \otimes m_{cd}) & \xrightarrow{\mathbf{a}_{m_{ab}, m_{bc}, m_{cd}}^{-1}} & (m_{ab} \otimes m_{bc}) \otimes m_{cd} \\ \text{id}_{m_{ab}} \otimes n_{bcd} \downarrow & & \downarrow n_{abc} \otimes \text{id}_{m_{cd}} \\ m_{ab} \otimes m_{bd} & \xrightarrow{n_{abd}} m_{ad} \xleftarrow{n_{acd}} & m_{ac} \otimes m_{cd} \end{array} \quad (3.13a)$$

¹¹cf. Definition 2.10

and

$$\begin{array}{ccc}
m_{ab} \otimes \text{id}_{e_b} & \xrightarrow{\text{id}_{m_{ab}} \otimes n_b} & m_{ab} \otimes m_{bb} \\
& \searrow l_{m_{ab}} & \downarrow n_{abb} \\
& & m_{ab}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\text{id}_{e_a} \otimes m_{ab} & \xrightarrow{n_a \otimes \text{id}_{m_{ab}}} & m_{aa} \otimes m_{ab} \\
& \searrow r_{m_{ab}} & \downarrow n_{aab} \\
& & m_{ab}
\end{array}
\quad (3.13b)$$

In formulæ, this reads as

$$n_{acd} \circ (n_{abc} \otimes \text{id}_{m_{cd}}) \circ a_{m_{ab}, m_{bc}, m_{cd}}^{-1} = n_{abd} \circ (\text{id}_{m_{ab}} \otimes n_{bcd}) \quad (3.14a)$$

and

$$n_{abb} \circ (\text{id}_{m_{ab}} \otimes n_b) = l_{m_{ab}} \quad \text{and} \quad n_{aab} \circ (n_a \otimes \text{id}_{m_{ab}}) = r_{m_{ab}} . \quad (3.14b)$$

Definition 3.9. A degree-2 Čech cocycle with values in a weak 2-group \mathcal{BG} is a degree-2 Čech cochain with values in \mathcal{BG} provided that it satisfies the equations (3.12) and (3.14). These equations are called the cocycle conditions of a weak principal 2-bundle and the degree-2 Čech cocycle is called its transition functions.

Pushing the analogy with the case of principle bundles further, we derive equivalence relations between weak principal 2-bundles from natural 2-transformations.

Definition 3.10. Any two weak principal 2-bundles are equivalent if and only if there is a weak natural 2-transformation between the corresponding weak 2-functors.

Explicitly, for weak principal 2-bundles Φ and $\tilde{\Phi}$, such a natural 2-transformation $\alpha : \tilde{\Phi} \rightarrow \Phi$ is given by the following data: we have 1-cells $\{m_a\}$ and 2-cells $\{n_{ab}\}$,

$$\begin{aligned}
m_a &: \tilde{e}_a \rightarrow e_a , \\
n_{ab} &: m_{ab} \otimes m_b \Rightarrow m_a \otimes \tilde{m}_{ab} ,
\end{aligned}
\quad (3.15)$$

such that the diagram

$$\begin{array}{ccc}
e_b & \xrightarrow{m_{ab}} & e_a \\
\uparrow m_b & \searrow n_{ab} & \uparrow m_a \\
\tilde{e}_b & \xrightarrow{\tilde{m}_{ab}} & \tilde{e}_a
\end{array}
\quad (3.16)$$

is commutative. The coherence conditions for natural 2-transformations require also the following diagrams

$$\begin{array}{ccc}
m_{ab} \otimes (m_b \otimes \tilde{m}_{bc}) & \xrightarrow{a_{m_{ab}, m_b, \tilde{m}_{bc}}^{-1}} & (m_{ab} \otimes m_b) \otimes \tilde{m}_{bc} \xrightarrow{n_{ab} \otimes \text{id}_{\tilde{m}_{bc}}} (m_a \otimes \tilde{m}_{ab}) \otimes \tilde{m}_{bc} \\
\uparrow \text{id}_{m_{ab}} \otimes n_{bc} & & \downarrow a_{m_a, \tilde{m}_{ab}, \tilde{m}_{bc}} \\
m_{ab} \otimes (m_{bc} \otimes m_c) & & m_a \otimes (\tilde{m}_{ab} \otimes \tilde{m}_{bc}) \\
\uparrow a_{m_{ab}, m_{bc}, m_c} & & \downarrow \text{id}_{m_a} \otimes \tilde{n}_{abc} \\
(m_{ab} \otimes m_{bc}) \otimes m_c & \xrightarrow{n_{abc} \otimes \text{id}_{m_c}} m_{ac} \otimes m_c \xrightarrow{n_{ac}} m_a \otimes \tilde{m}_{ac}
\end{array}
\quad (3.17a)$$

and

$$\begin{array}{ccc}
\mathrm{id}_{e_a} \otimes m_a & \xrightarrow{r_{m_a}} & m_a \xrightarrow{l_{m_a}^{-1}} m_a \otimes \mathrm{id}_{e_a} \\
\downarrow n_a \otimes \mathrm{id}_{m_a} & & \downarrow \mathrm{id}_{m_a} \otimes \tilde{n}_a \\
m_{aa} \otimes m_a & \xrightarrow{n_{aa}} & m_a \otimes \tilde{m}_{aa}
\end{array} \quad (3.17b)$$

to be commutative. In formulæ, this amounts to

$$\begin{aligned}
n_{ac} \circ (n_{abc} \otimes \mathrm{id}_{m_c}) &= (\mathrm{id}_{m_a} \otimes \tilde{n}_{abc}) \circ a_{m_a, \tilde{m}_{ab}, \tilde{m}_{bc}} \circ (n_{ab} \otimes \mathrm{id}_{\tilde{m}_{bc}}) \circ \\
&\quad \circ a_{m_{ab}, m_b, \tilde{m}_{bc}}^{-1} \circ (\mathrm{id}_{m_{ab}} \otimes n_{bc}) \circ a_{m_{ab}, m_{bc}, m_c}
\end{aligned} \quad (3.18a)$$

and

$$n_{aa} \circ (n_a \otimes \mathrm{id}_{m_a}) = (\mathrm{id}_{m_a} \otimes \tilde{n}_a) \circ l_{m_a}^{-1} \circ r_{m_a} . \quad (3.18b)$$

Definition 3.11. Any two degree-2 Čech cocycles $(\{m_{ab}\}, \{n_{abc}\}, \{n_a\})$ and $(\{\tilde{m}_{ab}\}, \{\tilde{n}_{abc}\}, \{\tilde{n}_a\})$ with values in a weak 2-group \mathcal{BG} are called equivalent if and only if there is a degree-1 Čech cochain $(\{m_a\}, \{n_{ab}\})$ with values in \mathcal{BG} such that the equations (3.15) and (3.18) are satisfied. These equations are called the coboundary conditions for a weak principal 2-bundle, and, slightly deviating from the usual nomenclature, the degree-1 Čech cochain $(\{m_a\}, \{n_{ab}\})$ is called a degree-2 Čech coboundary.

Definition 3.12. A weak principal 2-bundle that is equivalent to the weak principal 2-bundle specified by the functor

$$\{m_{ab} = \mathrm{id}_{e_a}\} , \quad \{n_{abc} = \mathrm{id}_{\mathrm{id}_{e_a}}\} , \quad \text{and} \quad \{n_a = \mathrm{id}_{\mathrm{id}_{e_a}}\} \quad (3.19)$$

is called trivial.

We shall give explicit formulæ in the case of trivial *semistrict* principal 2-bundles later on. For strict bundles, the 2-cells $\{n_a\}$ can always be chosen to be trivial, as was done, for instance, in [9]. The same is true here, as we shall verify now.

Lemma 3.13. Consider transition functions $(\{m_{ab}\}, \{n_{abc}\}, \{n_a\})$ of a weak principal 2-bundle Φ . The triple $(\{\tilde{m}_{ab}\}, \{\tilde{n}_{abc}\}, \{\tilde{n}_a\})$, which agrees with that of Φ except for

$$\{\tilde{m}_{aa} = \mathrm{id}_{e_a}\} , \quad \{\tilde{n}_{aab} = r_{\tilde{m}_{ab}}\} , \quad \{\tilde{n}_{abb} = l_{\tilde{m}_{ab}}\} \quad \text{and} \quad \{\tilde{n}_a = \mathrm{id}_{\mathrm{id}_{e_a}}\} , \quad (3.20)$$

defines another weak principal 2-bundle $\tilde{\Phi}$. In addition, these equations imply

$$\{\tilde{n}_{aaa} = r_{\mathrm{id}_{e_a}} = l_{\mathrm{id}_{e_a}}\} \quad (3.21)$$

Proof. One readily checks that the cocycle conditions (3.12) and (3.14) are readily satisfied for any possible doubling of indices. \square

Definition 3.14. For every weak principal 2-bundle Φ , the weak principal 2-bundle $\tilde{\Phi}$ obtained from the construction of Lemma 3.13 is called the normalisation of Φ .

Proposition 3.15. Every weak principal 2-bundle is equivalent to its normalisation.

Proof. The natural 2-transformation that yields the equivalence is given by

$$\{m_a = \text{id}_{e_a}\} \quad \text{and} \quad \left\{ n_{ab} = \begin{cases} r_{\tilde{m}_{ab}}^{-1} \circ l_{m_{ab}} & \text{for } a \neq b \\ n_a^{-1} \otimes \text{id}_{\text{id}_{e_a}} & \text{for } a = b \end{cases} \right\}. \quad (3.22)$$

One can readily verify that the coboundary conditions (3.15) and (3.18) are satisfied. \square

Corollary 3.16. Every weak principal 2-bundle is locally trivialisable.

Proof. By Proposition 3.15, a weak principal 2-bundle Φ is equivalent to its normalisation, for which we have

$$n_{aaa} = \text{id}_{\text{id}_e}, \quad n_a = \text{id}_{\text{id}_e}, \quad m_{aa} = \text{id}_e. \quad (3.23)$$

on any patch U_a . Thus, the weak principal 2-bundle is locally equivalent to a trivial one. \square

Recall that trivial principal bundles with structure group G are described by transition functions $\{g_{ab}\}$ of the form $g_{ab} = g_a g_b^{-1}$, where $\{g_a\}$ is a G -valued Čech 0-cochain. Note that the g_a can be multiplied by a G -valued function from the right, leaving $g_{ab} = g_a g_b^{-1}$ invariant. This is an equivalence relation, which is described by modifications in functorial language.

The corresponding equivalence relations are more comprehensive in the case of principal 2-bundles, as we shall see in the following. Consider two equivalent weak principal 2-bundles Φ and $\tilde{\Phi}$ with natural 2-transformations $\alpha : \tilde{\Phi} \rightarrow \Phi$ and $\tilde{\alpha} : \tilde{\Phi} \rightarrow \Phi$ between them. A weak 2-modification $\varphi : \alpha \Rightarrow \tilde{\alpha}$ is given by a map $\varphi : \alpha \rightarrow \tilde{\alpha}$ assumed to be smooth, which assigns to every object $(x, a) \in \check{C}(\mathcal{U})$ a 2-morphism $\varphi_{(x,a)} : \alpha_{(x,a)} \Rightarrow \tilde{\alpha}_{(x,a)}$. We set $o_a := \varphi_{(x,a)}$ so that $o_a : m_a \Rightarrow \tilde{m}_a$. Moreover, the following diagram is required to be commutative:

$$\begin{array}{ccc} m_{ab} \otimes m_b & \xrightarrow{\text{id}_{m_{ab}} \otimes o_b} & m_{ab} \otimes \tilde{m}_b \\ n_{ab} \Downarrow & & \Downarrow \tilde{n}_{ab} \\ m_a \otimes \tilde{m}_{ab} & \xrightarrow{o_a \otimes \text{id}_{\tilde{m}_{ab}}} & \tilde{m}_a \otimes \tilde{m}_{ab} \end{array} \quad (3.24)$$

that is,

$$\tilde{n}_{ab} \circ (\text{id}_{m_{ab}} \otimes o_b) = (o_a \otimes \text{id}_{\tilde{m}_{ab}}) \circ n_{ab}. \quad (3.25)$$

Definition 3.17. Any two degree-2 Čech coboundaries $(\{m_a\}, \{n_{ab}\})$ and $(\{\tilde{m}_a\}, \{\tilde{n}_{ab}\})$ with values in a weak Lie 2-group $\mathbf{B}\mathcal{G}$ between two degree-2 Čech cocycles $(\{m_{ab}\}, \{n_{abc}\}, \{n_a\})$ and $(\{\tilde{m}_{ab}\}, \{\tilde{n}_{abc}\}, \{\tilde{n}_a\})$ with values in $\mathbf{B}\mathcal{G}$ are said to be equivalent if and only if there is a degree-0 Čech cochain $\{o_a\}$ with values in $\mathbf{B}\mathcal{G}$ such that equations (3.18) are satisfied.

To define pullbacks and restrictions of principal 2-bundles, we proceed just as in the case of the functorial description of principal bundles; see Definitions 3.5 and 3.6. Recall that given a smooth map $\phi : X \rightarrow Y$ and a covering \mathfrak{U}_Y of Y , the pre-images of the patches in \mathfrak{U}_Y form a covering of \mathfrak{U}_X . The resulting groupoid morphisms $\check{\mathcal{C}}(\mathfrak{U}_X) \rightarrow \check{\mathcal{C}}(\mathfrak{U}_Y)$ can be extended trivially to a strict 2-functor $\phi_{\mathfrak{U}}$. We then give the following definitions.

Definition 3.18. The pullback of a weak principal 2-bundle Φ over Y with respect to an open cover \mathfrak{U}_Y along a map $\phi : X \rightarrow Y$ is the composition of 2-functors $\Phi \circ \phi_{\mathfrak{U}}$.

Definition 3.19. The restriction of a weak principal 2-bundle Φ over a manifold X to a submanifold Y inside X is the pullback of Φ along the embedding map $Y \hookrightarrow X$.

3.3. Semistrict and strict principal 2-bundles

We shall be specifically interested in weak principal 2-bundles with semistrict structure 2-groups. This implies a number of simplifications, which we discuss in the following.

Definition 3.20. A semistrict principal 2-bundle is a normalised weak principal 2-bundle with semistrict structure 2-group $\mathbf{B}\mathcal{G}$.

Explicitly, we have a weak 2-functor Φ described by a Čech 2-cocycle (transition functions) $(\{m_{ab}\}, \{n_{abc}\}, \{n_a\})$ with values in $\mathbf{B}\mathcal{G}$ such that

$$\{m_{aa} = \text{id}_{e_a}\}, \quad \{n_{aab} = \text{id}_{m_{ab}}\}, \quad \{n_{abb} = \text{id}_{m_{ab}}\}, \quad \text{and} \quad \{n_a = \text{id}_{\text{id}_{e_a}}\}. \quad (3.26a)$$

The cocycle conditions for this type of principal 2-bundle then read as

$$\begin{aligned} n_{abc} : m_{ab} \otimes m_{bc} &\Rightarrow m_{ac}, \\ n_{acd} \circ (n_{abc} \otimes \text{id}_{m_{cd}}) \circ \mathbf{a}_{m_{ab}, m_{bc}, m_{cd}}^{-1} &= n_{abd} \circ (\text{id}_{m_{ab}} \otimes n_{bdc}), \end{aligned} \quad (3.26b)$$

while the coboundary conditions and modifications are given by

$$\begin{aligned} m_a : \tilde{e}_a &\rightarrow e_a, \\ n_{ab} : m_{ab} \otimes m_b &\Rightarrow m_a \otimes \tilde{m}_{ab}, \\ n_{ac} \circ (n_{abc} \otimes \text{id}_{m_c}) &= (\text{id}_{m_a} \otimes \tilde{n}_{abc}) \circ \mathbf{a}_{m_a, \tilde{m}_{ab}, \tilde{m}_{bc}} \circ (n_{ab} \otimes \text{id}_{\tilde{m}_{bc}}) \circ \\ &\quad \circ \mathbf{a}_{m_{ab}, m_b, \tilde{m}_{bc}}^{-1} \circ (\text{id}_{m_{ab}} \otimes n_{bc}) \circ \mathbf{a}_{m_{ab}, m_{bc}, m_c} \end{aligned} \quad (3.26c)$$

and

$$\begin{aligned} o_a : m_a &\Rightarrow \tilde{m}_a , \\ \tilde{n}_{ab} \circ (\text{id}_{m_{ab}} \otimes o_b) &= (o_a \otimes \text{id}_{\tilde{m}_{ab}}) \circ n_{ab} , \end{aligned} \quad (3.26d)$$

respectively.

Remark 3.21. *A trivial semistrict principal 2-bundle is described by transition functions $(\{m_{ab}\}, \{n_{abc}\})$ given in terms of coboundary data $(\{m_a\}, \{n_{ab}\})$ according to*

$$\begin{aligned} m_a : \tilde{e}_a &\rightarrow e_a \quad \text{and} \quad n_{ab} : m_{ab} \otimes m_b \Rightarrow m_a , \\ n_{ac} \circ (n_{abc} \otimes \text{id}_{m_c}) &= n_{ab} \circ (\text{id}_{m_{ab}} \otimes n_{bc}) \circ \mathbf{a}_{m_{ab}, m_{bc}, m_c} , \end{aligned} \quad (3.27)$$

where $n_{aa} = \text{id}_{m_a}$.

To recover principal 2-bundles based on crossed modules as discussed in most of the current literature, we define the following.

Definition 3.22. *A strict principal 2-bundle is a weak principal 2-bundle with strict structure 2-group.*

A well-known result is then the following.

Proposition 3.23. *A strict principal 2-bundle with strict structure 2-group \mathcal{G} can be equivalently described by the following data based on a crossed module of Lie groups $(H \xrightarrow{\partial} G, \triangleright)$: a Čech 1-cochain $\{g_{ab}\}$ with values in G as well as a Čech 2-cochain $\{h_{abc}\}$ with values in H such that*

$$\partial(h_{abc})g_{ab}g_{bc} = g_{ac} \quad \text{and} \quad h_{acd}h_{abc} = h_{abd}(g_{ab} \triangleright h_{bcd}) . \quad (3.28a)$$

Analogously, coboundaries can be described in terms of Čech 0-cochains $\{g_a\}$ with values in G and Čech 1-cochains $\{h_{ab}\}$ with values in H . Two strict principal 2-bundles are equivalent if and only if

$$g_a \tilde{g}_{ab} = \partial(h_{ab})g_{ab}g_b \quad \text{and} \quad h_{ac}h_{abc} = (g_a \triangleright \tilde{h}_{abc})h_{ab}(g_{ab} \triangleright h_{bc}) . \quad (3.28b)$$

Finally, two coboundaries are equivalent if and only if there is a Čech 0-cochain $\{h_a\}$ with values in H such that

$$\tilde{g}_a = g_a \partial(h_a) \quad \text{and} \quad \tilde{h}_{ab} = (g_a \triangleright h_a h_b^{-1})h_{ab} . \quad (3.28c)$$

Proof: Let us again sketch the identification. For a strict principal 2-bundle, the cocycle and coboundary conditions, as well as the coherence equation for modifications, reduce to

$$\begin{aligned} n_{abc} : m_{ab} \otimes m_{bc} &\Rightarrow m_{ac} , \\ n_{acd} \circ (n_{abc} \otimes \text{id}_{m_{cd}}) &= n_{abd} \circ (\text{id}_{m_{ab}} \otimes n_{bcd}) , \end{aligned} \quad (3.29a)$$

and

$$\begin{aligned}
m_a &: \tilde{e}_a \rightarrow e_a , \\
n_{ab} &: m_{ab} \otimes m_b \Rightarrow m_a \otimes \tilde{m}_{ab} , \\
n_{ac} \circ (n_{abc} \otimes \text{id}_{m_c}) &= (\text{id}_{m_a} \otimes \tilde{n}_{abc}) \circ (n_{ab} \otimes \text{id}_{\tilde{m}_{bc}}) \circ (\text{id}_{m_{ab}} \otimes n_{bc}) ,
\end{aligned} \tag{3.29b}$$

and

$$\begin{aligned}
o_a &: m_a \Rightarrow \tilde{m}_a , \\
\tilde{n}_{ab} \circ (\text{id}_{m_{ab}} \otimes o_b) &= (o_a \otimes \text{id}_{\tilde{m}_{ab}}) \circ n_{ab} .
\end{aligned} \tag{3.29c}$$

Next, recall the identification of strict Lie 2-groups with crossed modules of Lie groups of Proposition 2.36. Assuming that the Lie 2-group \mathcal{G} equals $(\mathbf{H} \rtimes \mathbf{G}, \mathbf{G})$ in terms of the Lie groups \mathbf{G} and \mathbf{H} contained in the equivalent crossed module. We can then identify $m_{ab} = g_{ab}$ and $n_{abc} = (h_{abc}, g_{abc})$. From

$$\begin{aligned}
g_{abc} &= \mathbf{t}(n_{abc}) = g_{ac} = m_{ac} , \\
\mathbf{s}(n_{abc}) &= m_{ab} \otimes m_{bc} = g_{ab}g_{bc} = \partial(h_{abc}^{-1})g_{abc} ,
\end{aligned} \tag{3.30}$$

we immediately obtain the first equation in (3.28a). Likewise, using $\text{id}_{m_{ab}} = (\mathbb{1}_{\mathbf{H}}, g_{ab})$ and (2.19), it is a straightforward exercise to show that (3.29a) simplifies to the second equation in (3.28a).

Following the same line of arguments, the coboundary conditions (3.29b) and modifications (3.29c) are rewritten as (3.28b) and (3.28c). \square

Remark 3.24. *In the strict setting, we may define*

$$\ell_{ab} := n_{ab} \otimes \text{id}_{\overline{m}_b} , \tag{3.31}$$

where $m \otimes \overline{m} = \text{id}_e$. It is easy to see that $\ell_{ab} : m_{ab} \Rightarrow m_a \otimes \tilde{m}_{ab} \otimes \overline{m}_b$, and, in particular, if the bundle is trivial, then $\ell_{ab} : m_{ab} \Rightarrow m_a \otimes \overline{m}_b$. In this case, one may also show that n_{abc} can be rewritten in terms of ℓ_{ab} as

$$n_{abc} = \ell_{ac}^{-1} \circ (\ell_{ab} \otimes \ell_{bc}) . \tag{3.32}$$

It is amusing to note the resemblance with a trivial Abelian gerbe: the only difference is that ordinary products are replaced by \circ and \otimes .

In crossed module language, we may write

$$\ell_{ab} = n_{ab} \otimes \text{id}_{\overline{m}_b} = (h_{ab}, g_{ab}g_b) \otimes (\mathbb{1}_{\mathbf{H}}, g_b^{-1}) = (h_{ab}, g_{ab}) \tag{3.33}$$

with $\mathfrak{s}(\ell_{ab}) = g_{ab}$ and $\mathfrak{t}(\ell_{ab}) = \mathfrak{t}(h_{ab})g_{ab} \stackrel{!}{=} g_ag_b^{-1}$ and hence, $g_{ab} = \mathfrak{t}(h_{ab}^{-1})g_ag_b^{-1}$ in agreement with the previous discussion for trivial bundles. Furthermore,

$$\begin{aligned} n_{abc} &= (h_{abc}, g_{abc}) = \ell_{ac}^{-1} \circ (\ell_{ab} \otimes \ell_{bc}) \\ &= (h_{ac}^{-1}, \mathfrak{t}(h_{ac})g_{ac}) \circ (h_{ab}(g_{ab} \triangleright h_{bc}), g_{ab}g_{bc}) \\ &= (h_{ac}^{-1}h_{ab}(g_{ab} \triangleright h_{bc}), g_{ab}g_{bc}) \end{aligned} \quad (3.34)$$

and hence, $h_{abc} = h_{ac}^{-1}h_{ab}(g_{ab} \triangleright h_{bc})$ in agreement with the previous discussion for trivial bundles. Note that $g_{abc} = g_{ab}g_{bc}$ as derived previously. Note also that we have used $\mathfrak{t}(\ell_{ab}) = \mathfrak{t}(h_{ab})g_{ab} = g_ag_b^{-1}$ from our previous discussion.

4. Differentiating semistrict Lie 2-groups

To define connective structures on semistrict principal 2-bundles, we first need to differentiate a semistrict Lie 2-group to a semistrict Lie 2-algebra. We do this following an idea of Ševera [17], see also Jurčo [33]. This method requires a description of semistrict principal 2-bundles in terms of generalised Čech cocycles, which we developed in the previous section.

4.1. Basic ideas

Consider a surjective submersion $\sigma : Y \rightarrow X$ for two smooth manifolds X and Y and a Lie group \mathbf{G} . We denote the sheaf of smooth differential p -forms on X by Ω_X^p and set $\Omega_X^\bullet := \bigoplus_p \Omega_X^p$. Furthermore, let $\mathbb{R}^{0|1} := \Pi\mathbb{R}$, where Π is the Graßmann-parity changing functor.

Definition 4.1. Descent data on a surjective submersion $\sigma : Y \rightarrow X$ with values in \mathbf{G} describes the descent of a trivial principal bundle with structure group \mathbf{G} on Y to a non-trivial such bundle on X . It is given by a non-Abelian Čech 1-cocycle relative to the cover provided by Y .

In particular, for the obvious projection $\sigma : \mathbb{R}^{0|1} \times X \rightarrow X$, we have maps $g : \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \times X \rightarrow \mathbf{G}$ such that

$$g(\theta_0, \theta_1, x)g(\theta_1, \theta_2, x) = g(\theta_0, \theta_2, x) , \quad (4.1)$$

which implies

$$g(\theta_0, \theta_0, x) = \mathbb{1}_{\mathbf{G}} \quad \text{and} \quad g(\theta_0, \theta_1, x) = g(\theta_1, \theta_0, x)^{-1} . \quad (4.2)$$

Note that the maps $\{g(\theta_0, \theta_1, x)\}$ satisfy the cocycle conditions for a principal bundle (3.1), where the patches are indexed by elements of $\mathbb{R}^{0|1}$.

Denote by $\Delta(X)$ the set of all descent data on $\sigma : \mathbb{R}^{0|1} \times X \rightarrow X$ with values in G . Then Δ is in fact a contravariant functor from the category of manifolds to the category of sets. This functor is representable by $\mathfrak{g}[1]$, where \mathfrak{g} is the Lie algebra of G and $\mathfrak{g}[1] := \Pi\mathfrak{g}$ is the Lie algebra shifted by one degree in parity.

Calculating the moduli of Δ , which can be readily done, yields the Lie algebra of G as a vector space. To describe its Lie bracket, we compute the action of its Chevalley–Eilenberg differential¹² d_{CE} . This differential is governed by a generator of the natural action of $C^\infty(\mathbb{R}^{0|1}, \mathbb{R}^{0,1})$ on the descent data.

The latter idea goes back to work by Kontsevich [34], see also [17], which we review in more detail in the following.

Proposition 4.2. *There is an isomorphism $H^0(X, \Omega_X^\bullet) \cong C^\infty(C^\infty(\mathbb{R}^{0|1}, X), \mathbb{R})$.*

Proof. Consider first the simple example $X = \mathbb{R}^n$ with coordinates (x^1, \dots, x^n) . An element of $C^\infty(\mathbb{R}^{0|1}, X)$ is parameterised as $(x^1, \dots, x^n) = (a^1 + \alpha^1\theta, \dots, a^n + \alpha^n\theta)$, where $\theta, \alpha^i \in \mathbb{R}^{0|1}$ are Grassmann-odd and $a^i \in \mathbb{R}$ are Grassmann even. We thus have $C^\infty(\mathbb{R}^{0|1}, X) \cong \mathbb{R}^{n|n}$. Functions on $\mathbb{R}^{n|n}$ are polynomial in the odd variables and by identifying a^i with the coordinates on X and α^i with the corresponding differential 1-forms, we obtain $C^\infty(\mathbb{R}^{n|n}, \mathbb{R}) \cong H^0(X, \Omega_X^\bullet)$.

For a general smooth manifold X , we have correspondingly a local isomorphism between $C^\infty(\mathbb{R}^{0|1}, X)$ and $TX[1] := \Pi TX$. This local isomorphism can be glued to a global one, as it is invariant under change of coordinates. \square

In the following, we shall call the coefficients in the Taylor series expansion of $f \in C^\infty(C^\infty(\mathbb{R}^{0|1}, X), \mathbb{R})$ in the Grassmann-odd coordinate the components of f .

The de Rham differential d on $H^0(X, \Omega_X^\bullet)$ follows from the action of $C^\infty(\mathbb{R}^{0|1}, \mathbb{R}^{0,1})$ on $C^\infty(\mathbb{R}^{0|1}, X)$. Concretely, transformations of the form $\theta \mapsto \tilde{\theta} = b\theta + \beta$ for $b \in \mathbb{R}$, $\beta \in \mathbb{R}^{0|1}$ induce an action on maps $x \in C^\infty(\mathbb{R}^{0|1}, X)$ which is given locally by coordinate functions

$$x^i(\theta) = a^i + \alpha^i\theta \mapsto x^i(\tilde{\theta}) = a^i + (b\theta + \beta)\alpha^i = (a^i + \beta\alpha^i) + b\alpha^i\theta. \quad (4.3)$$

Translated into differential forms, this means that $x^i \mapsto x^i + \beta dx^i$ and $dx^i \mapsto b dx^i$. We thus arrive at the following result.

Proposition 4.3. *The action of the de Rham differential d on $H^0(X, \Omega_X^\bullet)$ translates to the following action of the generator d_K of $C^\infty(\mathbb{R}^{0|1}, \mathbb{R}^{0,1})$*

$$d_K f(x(\theta)) := \frac{d}{d\varepsilon} f(x(\theta + \varepsilon)) \quad (4.4)$$

for any $f \in C^\infty(C^\infty(\mathbb{R}^{0|1}, X), \mathbb{R})$.

¹²see Appendix A for the relevant definitions

The differential d_K extends to smooth functions $f \in C^\infty(C^\infty(\mathbb{R}^{0|k}, X), \mathbb{R})$. Specifically, its action on a function of several Grassmann-odd coordinates $(\theta_1, \dots, \theta_k)$ is defined diagonally,

$$d_K f(x(\theta_1, \dots, \theta_k)) := \frac{d}{d\varepsilon} f(x(\theta_1 + \varepsilon, \dots, \theta_k + \varepsilon)) . \quad (4.5)$$

For example, for a Grassmann-even function $f(x(\theta_1, \theta_2)) = f_0 + \phi_1 \theta_1 + \phi_2 \theta_2 + F \theta_1 \theta_2$, we obtain

$$d_K f(x(\theta_1, \theta_2)) = \frac{d}{d\varepsilon} f(x(\theta_1 + \varepsilon, \theta_2 + \varepsilon)) = -\phi_1 - \phi_2 + (\theta_1 - \theta_2)F . \quad (4.6)$$

Comparing coefficients, we read off the action of an induced operator, again denoted by d_K on the components

$$d_K f_0 = -\phi_1 - \phi_2 , \quad d_K \phi_1 = F , \quad d_K \phi_2 = -F , \quad \text{and} \quad d_K F = 0 . \quad (4.7)$$

Proposition 4.4. *The operator d_K has the following properties:*

- (i) $d_K \circ d_K = 0$,
- (ii) for any $f \in C^\infty(C^\infty(\mathbb{R}^{0|k}, X), \mathbb{R})$ and $g \in C^\infty(C^\infty(\mathbb{R}^{0|l}, X), \mathbb{R})$, the operator d_K obeys a graded Leibniz rule,

$$d_K(fg) = (d_K f)g + (-1)^{|f|} f d_K g , \quad (4.8)$$

where $|f|$ denotes the Grassmann parity of f .

In turn, the Leibniz rule (ii) induces a Leibniz rule for the induced operator on the components.

Proof. Property (i) follows immediately from the definition. To prove property (ii), it is sufficient to consider functions $f = a + b\theta_0$ and $g = c + d\theta_1$. Equation (4.8) is then readily verified. \square

Since d_K obeys a Leibniz rule and since $d_K \circ d_K = 0$, we shall call it a differential in the following.

Proposition 4.5. *The differential d_K induces a complex*

$$\mathcal{S}_X \xrightarrow{d_K} TX[1] \xrightarrow{d_K} TX[2] \xrightarrow{d_K} \dots , \quad (4.9)$$

where \mathcal{S}_X is the sheaf of smooth functions on X and $TX[p]$ is the tangent sheaf with the stalks of elements of degree p .

4.2. Lie algebra of a Lie group

Having collected all relevant ideas, let us put them to use and compute the Lie algebra of a Lie group. This has been done in [17] and [33], and our discussion below is an expanded version of the one found in these references.

Consider a finite-dimensional Lie group G with Lie algebra $T_{\mathbb{1}_G}G = \mathfrak{g}$. To prepare our discussion for Lie 2-groups, we shall not assume that G is a matrix group. Instead, we rely on the fact that there is a diffeomorphism φ between a small neighbourhood $U_{\mathfrak{g}}$ of 0 in $\mathfrak{g} := T_{\mathbb{1}_G}G$ and a small neighbourhood U_G of $\mathbb{1}_G$ in G . That is, we have $\varphi(a) = g$, where $a \in U_{\mathfrak{g}}$ and $g \in U_G$ with $\varphi(0) = \text{id}_{\mathbb{1}_G}$. If G is a matrix group, one might suggestively write $g = \varphi(a) = \mathbb{1}_G + a$.

We now restrict ourselves to infinitesimal neighbourhoods by considering elements of $\mathfrak{g}[1]$ multiplied by some Graßmann-odd coordinates θ_i . We need a rule for multiplying group elements corresponding to infinitesimal algebra elements:

Proposition 4.6. *For $a, b \in \mathfrak{g}[1]$, we have the following relations:*

$$\varphi(a\theta_0)^{-1} = \varphi(-a\theta_0) \quad \text{and} \quad \varphi^{-1}(\varphi(a\theta_0)\varphi(b\theta_1)) = a\theta_0 + b\theta_1 + \psi_2(a, b)\theta_0\theta_1, \quad (4.10)$$

or, more suggestively for a matrix group G ,

$$(\mathbb{1}_G + a\theta_0)^{-1} = \mathbb{1}_G - a\theta_0 \quad \text{and} \quad (\mathbb{1}_G + a\theta_0)(\mathbb{1}_G + b\theta_1) = \mathbb{1}_G + a\theta_0 + b\theta_1 + \psi_2(a, b)\theta_0\theta_1, \quad (4.11)$$

where $\psi_2(a, a) = -\frac{1}{2}[a, a]$.

Proof. First of all, it is clear that $\varphi^{-1}(\varphi(a\theta_0)\varphi(b\theta_1))$ is a polynomial in the Graßmann variables. The terms linear in θ_0 and θ_1 of this expression then follow from putting θ_1 or θ_0 to zero, respectively. In the special case, $\theta_1 = \theta_0$ and $b = -a$, we recover the first equation of (4.10). It remains to understand the term $\psi_2(a, b)$. For this, consider the expression

$$\begin{aligned} \xi(a, b, c, d) &:= \varphi^{-1}(\varphi(-a\theta_0)\varphi(-b\theta_1)\varphi(c\theta_0)\varphi(d\theta_1)) \\ &= (c - a)\theta_0 + (d - b)\theta_1 - (\psi_2(a, b) - \psi_2(a, d) + \psi_2(b, c) + \psi_2(c, d))\theta_0\theta_1, \end{aligned} \quad (4.12)$$

where the expansion follows from considering all cases in which one or more of the a, b, c, d vanish. Note that $\xi(a, b, a, b)$ is the algebra element corresponding to the group commutator, which implies that $[a, b] = -\psi_2(a, b) - \psi_2(b, a)$. \square

Analogously, one can define products between elements g and a of G and $\mathfrak{g}[1] = \Pi\text{Lie}(G)$, respectively. For matrix groups, we simply write ga . For more general groups, one replaces such expressions by the pullback g_*a of a along the map $g : \mathbb{1}_G \mapsto g$.

Consider now descent data on $\mathbb{R}^{0|1} \times X \rightarrow X$, i.e. functions $g : \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \times X \rightarrow \mathbf{G}$ satisfying the cocycle condition

$$g(\theta_0, \theta_1, x)g(\theta_1, \theta_2, x) = g(\theta_0, \theta_2, x) . \quad (4.13)$$

As we want to regard g as a functor from X to descent data, we suppress the dependence on x in the following. Note that the cocycle condition implies $g(\theta_0, \theta_0) = \mathbb{1}$ as well as $g(0, \theta_0)^{-1} = g(\theta_0, 0)$. We can then read off a trivialising coboundary:

Lemma 4.7. *The \mathbf{G} -valued functions $g(\theta_0) : \mathbb{R}^{0|1} \rightarrow \mathbf{G}$ with $g(\theta_0) = g(0, \theta_0)$ satisfy the equations*

$$g(\theta_0, \theta_1) = g(\theta_0)g(\theta_0)^{-1} . \quad (4.14)$$

We can expand¹³ $g(\theta_0) = \mathbb{1}_{\mathbf{G}} + a\theta_0$ for some $a \in \mathfrak{g}[1]$, because $g(0) = g(0, 0) = \mathbb{1}_{\mathbf{G}}$. Together with Propositions 4.3 and 4.6, we arrive at the following Lemma:

Lemma 4.8. *We have*

$$g(\theta_0, \theta_1) = (\mathbb{1}_{\mathbf{G}} + a\theta_0)(\mathbb{1}_{\mathbf{G}} - a\theta_1) = \mathbb{1}_{\mathbf{G}} + a(\theta_0 - \theta_1) + \frac{1}{2}[a, a]\theta_0\theta_1 \quad (4.15)$$

with the induced differential

$$d_K a + \frac{1}{2}[a, a] = 0 . \quad (4.16)$$

As stated above, we wish to identify the induced action of the differential d_K with the Chevalley–Eilenberg differential d_{CE} on \mathfrak{g} . Recall that the Chevalley–Eilenberg differential of a Lie algebra \mathfrak{g} acts as

$$d_{\text{CE}} \check{\tau}^i = -\frac{1}{2}f_{jk}^i \check{\tau}^j \otimes \check{\tau}^k , \quad (4.17)$$

on elements of \mathfrak{g}^\vee , where $\check{\tau}^i$ is some basis of \mathfrak{g}^\vee and f_{jk}^i are the structure constants of \mathfrak{g} with respect to the dual basis τ_i of $\mathfrak{g}[1]$ with $\langle \check{\tau}^i, \tau_j \rangle = \delta_j^i$. Equation (4.16) amounts to the Maurer–Cartan equation $d_{\text{CE}} a + \frac{1}{2}[a, a] = 0$, which should be regarded as equation (4.17) evaluated as a polynomial in a^i , where $a = a^i \tau_i$.

Altogether, we have proved the following theorem.

Theorem 4.9. *The functor from manifolds X to \mathbf{G} -valued descent data on surjective submersions $\mathbb{R}^{0|1} \times X \rightarrow X$ is parameterised by elements of $\mathfrak{g}[1] = \text{ILie}(\mathbf{G})$ and the action of the differential d_K on this descent data is given by the action of the Chevalley–Eilenberg differential corresponding to \mathfrak{g} .*

¹³To simplify notation, we use suggestive notation for matrix groups, which is readily translated to general expressions involving the diffeomorphism φ .

Finally, let us consider equivalence relations on the descent data $g(\theta_0, \theta_1)$ parameterised by a function $p : \mathbb{R}^{0|1} \rightarrow \mathbf{G}$ with $p(\theta) = p + \pi\theta$ for some $p \in \mathbf{G}$ and $\pi \in \mathfrak{g}[1]$ according to

$$\tilde{g}(\theta_0, \theta_1) = p(\theta_0)g(\theta_0, \theta_1)p(\theta_1)^{-1} = \mathbb{1}_{\mathbf{G}} + (pap^{-1} + \pi p^{-1})(\theta_0 - \theta_1) + \cdots, \quad (4.18)$$

where the ellipsis denotes terms proportional to $\theta_0\theta_1$. Together with the induced differential $d_K p = -\pi$, we obtain the following.

Proposition 4.10. *Under equivalence transformations of the descent data parameterised by $p(\theta)$, we have*

$$\tilde{a} = pap^{-1} + \pi p^{-1} = pap^{-1} + p d_K p^{-1}. \quad (4.19)$$

The equation $d_K a + \frac{1}{2}[a, a] = 0$ is invariant under such transformations.

Note that by replacing d_K by the de Rham differential in all of the above, we recover the definition of the curvature of a connection 1-form on a principal bundle with structure group \mathbf{G} as well as its gauge transformation. We will make use of this observation later on.

4.3. Semistrict Lie 2-algebra of a semistrict Lie 2-group

We now generalize the previous discussion to the case of semistrict Lie 2-groups $\mathcal{G} = (M, N)$, which we regard again as particular weak Lie 2-groupoid $\mathcal{B}\mathcal{G} = (\{e\}, M, N)$. Here, the diffeomorphism $\varphi = (\varphi_M, \varphi_N)$ goes between neighbourhoods U_M of id_e and U_N of id_{id_e} as well as neighbourhoods $U_{\mathfrak{m}}$ of $\mathfrak{m} := T_{\text{id}_e} M$ and $U_{\mathfrak{n}}$ of $\mathfrak{n} := T_{\text{id}_{\text{id}_e}} N$. We shall write suggestively $\text{id}_e + a\theta$ and $\text{id}_{\text{id}_e} + b\theta$ for $\varphi_M(a\theta)$ and $\varphi_N(b\theta)$, where $a \in \mathfrak{m}[1]$ and $b \in \mathfrak{n}[1]$.

Definition 4.11. *We implicitly define products \otimes on $\mathfrak{m}[1]$ and $\mathfrak{n}[1]$ according to*

$$(\text{id} + a_1\theta_1) \otimes (\text{id} + a_2\theta_2) =: \text{id} + a_1\theta_1 + a_2\theta_2 - (a_1 \otimes a_2)\theta_1\theta_2, \quad (4.20)$$

where id is id_e and id_{id_e} and $a_{1,2}$ are elements of $\mathfrak{m}[1]$ and $\mathfrak{n}[1]$, respectively.

Note that the linear terms on the right-hand side of (4.20) are fixed by considering the cases $a_1 = 0$ and $a_2 = 0$.

Proposition 4.12. *The products \otimes on $\mathfrak{m}[1]$ and $\mathfrak{n}[1]$ are bilinear.*

Proof. For more general elements of \mathfrak{m} and \mathfrak{n} , we have the following expansion of the product \otimes :

$$\begin{aligned} (\text{id} + a_1\theta_1 + a_2\theta_2) \otimes (\text{id} + a_3\theta_3 + a_4\theta_4) &= \text{id} + a_1\theta_1 + a_2\theta_2 + a_3\theta_3 + a_4\theta_4 - \\ &\quad - (a_1 \otimes a_3)\theta_1\theta_3 - (a_2 \otimes a_3)\theta_2\theta_3 - (a_1 \otimes a_4)\theta_1\theta_4 - (a_2 \otimes a_4)\theta_2\theta_4 + \mathcal{O}(\theta^3), \end{aligned} \quad (4.21)$$

where $a_{1,2,3,4}$ are elements of $\mathfrak{m}[1]$ or $\mathfrak{n}[1]$. Here, the expansion is fixed by Definition 4.11 and considering the special cases $a_1 = a_2 = 0$ and $a_3 = a_4 = 0$. Linearity follows from (4.21) by putting $\theta_2 = \theta_1$ and $\theta_4 = \theta_3$. \square

We now turn to the maps induced by the structure maps \mathbf{s}, \mathbf{t} and id on $\mathfrak{n}[1]$ and $\mathfrak{m}[1]$.

Remark 4.13. For elements $a \in \mathfrak{m}[1]$ and $b \in \mathfrak{n}[1]$, we have

$$\begin{aligned} \mathbf{s}(\text{id}_{\text{id}_e} + b\theta) &=: \text{id}_e + \text{ds}_0(b)\theta, & \mathbf{t}(\text{id}_{\text{id}_e} + b\theta) &=: \text{id}_e + \text{dt}_0(b)\theta, \\ \text{id}_{\text{id}_e + a\theta} &=: \text{id}_{\text{id}_e} + \text{d id}_0(a)\theta, \end{aligned} \quad (4.22)$$

where a subscript 0 indicates that the differential is to be taken at id_{id_e} or id_e .

For more general elements of \mathfrak{n} and \mathfrak{m} , the following result holds.

Proposition 4.14. Around $\text{id}_e + a_1\theta_1 + a_2\theta_2$ and $\text{id}_{\text{id}_e} + b_1\theta_1 + b_2\theta_2$ for some $a_{1,2} \in \mathfrak{m}[1]$ and $b_{1,2} \in \mathfrak{n}[1]$, the structure maps expand as follows:

$$\begin{aligned} \text{id}_{\text{id}_e + a_1\theta_1 + a_2\theta_2} &= \\ &= \text{id}_{\text{id}_e} + \text{d id}_0(a_1)\theta_1 + \text{d id}_0(a_2)\theta_2 - (\text{d id}_0(a_1 \otimes a_2) - \text{d id}_0(a_1) \otimes \text{d id}_0(a_2))\theta_1\theta_2, \\ \mathbf{s}(\text{id}_{\text{id}_e} + b_1\theta_1 + b_2\theta_2) &= \\ &= \text{id}_e + \text{ds}_0(b_1)\theta_1 + \text{ds}_0(b_2)\theta_2 - (\text{ds}_0(b_1 \otimes b_2) - \text{ds}_0(b_1) \otimes \text{ds}_0(b_2))\theta_1\theta_2, \\ \mathbf{t}(\text{id}_{\text{id}_e} + b_1\theta_1 + b_2\theta_2) &= \\ &= \text{id}_e + \text{dt}_0(b_1)\theta_1 + \text{dt}_0(b_2)\theta_2 - (\text{dt}_0(b_1 \otimes b_2) - \text{dt}_0(b_1) \otimes \text{dt}_0(b_2))\theta_1\theta_2. \end{aligned} \quad (4.23)$$

Proof. The map id is compatible with the product \otimes on M in the following way:

$$\text{id}_{(\text{id}_e + a_1\theta_1) \otimes (\text{id}_e + b_2\theta_2)} = \text{id}_{\text{id}_e + b_1\theta_1} \otimes \text{id}_{\text{id}_e + b_2\theta_2}. \quad (4.24)$$

Expanding both sides of the equation according to our previous definitions yields the desired result. The argument for the maps \mathbf{s} and \mathbf{t} is fully analogous. \square

Finally, we have to discuss an induced concatenation map on \mathfrak{n} . Note that if $\text{ds}_0(b_1) = \text{dt}_0(b_2)$ for some $b_{1,2} \in \mathfrak{n}[1]$, then $\mathbf{s}(\text{id}_{\text{id}_e} + b_1\theta) = \mathbf{t}(\text{id}_{\text{id}_e} + b_2\theta)$.

Definition 4.15. For elements $b_{1,2}$ of $\mathfrak{n}[1]$ with $\text{ds}_0(b_1) = \text{dt}_0(b_2)$, we define implicitly

$$(\text{id}_{\text{id}_e} + b_1\theta) \circ (\text{id}_{\text{id}_e} + b_2\theta) =: \text{id}_{\text{id}_e} + b_1 \circ b_2\theta. \quad (4.25)$$

It trivially follows that $b_1 \circ 0 = b_1$ for $\text{ds}_0(b_1) = 0$ and $0 \circ b_2 = b_2$ for $\text{dt}_0(b_2) = 0$. More generally, the induced concatenation map satisfies the following.

Proposition 4.16. *For $b_{1,2,3,4} \in \mathfrak{n}[1]$ with*

$$\mathrm{ds}_0(b_1) = \mathrm{dt}_0(b_3), \quad \mathrm{ds}_0(b_2) = \mathrm{dt}_0(b_4), \quad \mathrm{ds}_0(b_1 \otimes b_2) = \mathrm{dt}_0(b_3 \otimes b_4), \quad (4.26)$$

we have

$$(\mathrm{id}_{\mathrm{id}_e} + b_1\theta_1 + b_2\theta_2) \circ (\mathrm{id}_{\mathrm{id}_e} + b_3\theta_1 + b_4\theta_2) = \mathrm{id}_{\mathrm{id}_e} + b_1 \circ b_3\theta_1 + b_2 \circ b_4\theta_2. \quad (4.27)$$

Remark 4.17. *Note that above, we linearised all the structure maps $\mathfrak{s}, \mathfrak{t}, \mathrm{id}, \otimes, \circ$ at id_e or $\mathrm{id}_{\mathrm{id}_e}$ and obtained maps on $\mathfrak{m}[1]$ or $\mathfrak{n}[1]$. We can certainly consider linearisations also at other points p of M or N , leading to maps on $T_p M[1]$ or $T_p N[1]$. The formulas in these cases are obvious generalisations of the ones derived above.*

Remark 4.18. *In the following, we shall simply write $\mathfrak{s}, \mathfrak{t}$ and id for $\mathrm{ds}_0, \mathrm{dt}_0$ and $\mathrm{id}_{\mathrm{id}_e}$, slightly abusing notation. The distinction with the finite maps on M and N should always be clear from the context.*

We are now ready to discuss the differentiation of the semistrict Lie 2-group $\mathcal{B}\mathcal{G} = (\{e\}, M, N)$ to a semistrict Lie 2-algebra. Let X be a smooth manifold with covering $\mathfrak{U} = \{U_a\}$. Consider a trivial semistrict principal 2-bundle Φ over X with semistrict structure 2-group $\mathcal{B}\mathcal{G}$. It is characterised by a Čech 2-cocycle $(\{m_{ab}\}, \{n_{abc}\}, \{n_a\})$ with values in $\mathcal{B}\mathcal{G}$ which is cohomologous to the trivial one, cf. Remark 3.21.

Following our discussion for Lie groups, we consider again a functor from X to descent data on the surjective submersion $\mathbb{R}^{0|1} \times X \rightarrow X$. This functor is represented by 1-cells $\{m_{01} := m(\theta_0, \theta_1)\}$ and 2-cells $\{n_{012} := n(\theta_0, \theta_1, \theta_2)\}$ so that

$$n_{012} : m_{01} \otimes m_{12} \Rightarrow m_{02}, \quad (4.28a)$$

and

$$n_{023} \circ (n_{012} \otimes \mathrm{id}_{m_{23}}) = n_{013} \circ (\mathrm{id}_{m_{12}} \otimes n_{123}) \circ \mathfrak{a}_{m_{01}, m_{12}, m_{23}}. \quad (4.28b)$$

As we saw in Lemma 4.7, principal bundles obtained from descent data on $\mathbb{R}^{0|1} \times X \rightarrow X$ are necessarily trivial. The analogous statement in the case of semistrict principal 2-bundles is the following.

Lemma 4.19. *The functor $(\{m_{01}\}, \{n_{012}\})$ is trivialised by the following Čech 1-cochains $(\{m_0\}, \{n_{01}\})$:*

$$m_0 := m(\theta_0) := m(\theta_0, 0) \quad \text{and} \quad n_{01} := n(\theta_0, \theta_1) = n(\theta_0, \theta_1, 0). \quad (4.29)$$

That is, $n_{01} : m_{01} \otimes m_1 \Rightarrow m_0$ with

$$n_{02} \circ (n_{012} \otimes \mathrm{id}_{m_2}) = n_{01} \circ (\mathrm{id}_{m_{12}} \otimes n_{12}) \circ \mathfrak{a}_{m_{01}, m_{12}, m_2}. \quad (4.30)$$

Proof. This statement is readily proved by computation, where we use $\mathbf{a}_{m,m',\text{id}_e}$ is trivial for all $m, m' \in M$ (see Proposition 2.28). \square

Remark 4.20. *There is clearly a one-to-one correspondence between functors given by the descent data $(\{m_{01}\}, \{n_{012}\})$ and trivialising Čech 1-cochains $(\{m_0\}, \{n_{01}\})$. Moreover, by a modification isomorphism, any trivialising Čech 1-cochain $(\{m_0\}, \{n_{01}\})$ is equivalent to one of the form (4.29).*

Recall that semistrict principal 2-bundles are described by normalised cocycles, which implies

$$m(0) = \text{id}_e \quad \text{and} \quad n(\theta_0, 0) = \text{id}_{m_0} , \quad (4.31)$$

cf. Lemma 3.13. This leads us to the following result.

Proposition 4.21. *Descent data $(\{m_{01}\}, \{n_{012}\})$ and the corresponding coboundary data $(\{m_0\}, \{n_{01}\})$ are parametrised by 1-cells $\alpha \in \mathbf{m}[1]$ and 2-cells $\beta \in \mathbf{n}[2]$ with*

$$\alpha : 0 \rightarrow 0 \quad \text{and} \quad \beta : \mathbf{s}(\beta) \Rightarrow 0 \quad (4.32)$$

according to the following expansions in the Graßmann-odd coordinates:

$$m_0 = \text{id}_e + \alpha\theta_0 , \quad (4.33a)$$

$$n_{01} = \text{id}_{\text{id}_e} + \text{id}_\alpha\theta_0 + \beta\theta_0\theta_1 , \quad (4.33b)$$

$$m_{01} = \text{id}_e + \alpha(\theta_0 - \theta_1) + [\alpha \otimes \alpha + \mathbf{s}(\beta)]\theta_0\theta_1 , \quad (4.33c)$$

$$\begin{aligned} n_{012} = & \text{id}_{\text{id}_e} + \text{id}_\alpha(\theta_0 - \theta_2) + \beta(\theta_0\theta_1 + \theta_1\theta_2 - \theta_0\theta_2) + \\ & + \text{id}_{\alpha \otimes \alpha + \mathbf{s}(\beta)}\theta_0\theta_2 + [\text{id}_\alpha \otimes \beta - \beta \otimes \text{id}_\alpha + \mu(\alpha, \alpha, \alpha)]\theta_0\theta_1\theta_2 . \end{aligned} \quad (4.33d)$$

Proof. The expansion of m_0 is a direct consequence of (4.31) while the expansion (4.33b) follows directly from the conditions $n_{00} = \text{id}_{m_0} = \text{id}_{\text{id}_e} + \theta_0\text{id}_\alpha$, $n(\theta_0) = \text{id}_{m_0}$, and $\mathbf{t}(n_{01}) = m_0 = \text{id}_e + \theta_0\alpha$. The expansion (4.33c) follows from (4.33b) by comparing coefficients in $\mathbf{s}(n_{01}) = m_{01} \otimes m_1$, where we used the identity

$$\begin{aligned} (\text{id}_e + \alpha(\theta_0 - \theta_2) + \alpha_2\theta_0\theta_2) \otimes (\text{id}_e + \alpha\theta_2) = \\ = (\text{id}_e + (\alpha - \tfrac{1}{2}\alpha_2(\theta_0 + \theta_2))(\theta_0 - \theta_2)) \otimes (\text{id}_e + \alpha\theta_2) \\ = \text{id}_e + \alpha\theta_0 + (\alpha_2 - \alpha \otimes \alpha)\theta_0\theta_2 \end{aligned} \quad (4.34)$$

to evaluate the product.

To derive the expansion (4.33d), we use $n(\theta_0, \theta_1, 0) = n(\theta_0, \theta_1)$ together with the normalisation $n_{001} = \text{id}_{m_{01}}$ and $n_{011} = \text{id}_{m_{01}}$. To find the term of cubic order in (4.33d) from (4.30) and (4.33a)–(4.33c), we require an expansion of the associator $\mathbf{a}_{m_{01}, m_{12}, m_2}$. Since

according to Proposition 2.28 $\mathbf{a}_{\text{id}_e, m, m'}$, $\mathbf{a}_{m, \text{id}_e, m'}$, and $\mathbf{a}_{m, m', \text{id}_e}$ are trivial for all $m, m' \in M$, we can write

$$\mathbf{a}_{m_{01}, m_{12}, m_2} =: \text{id}_{m_{01} \otimes (m_{12} \otimes m_2)} + \mu(\alpha, \alpha, \alpha) \theta_0 \theta_1 \theta_2, \quad (4.35)$$

defining a linearised 2-cell $\mu(\alpha, \alpha, \alpha) : \alpha \otimes (\alpha \otimes \alpha) - (\alpha \otimes \alpha) \otimes \alpha \Rightarrow 0$. In order to evaluate (4.30) for coboundaries given in (4.33), we note that $n_{012} = \text{id}_{\text{id}_e} + \nu_1 \theta_0 + \nu_2 \theta_2$ and $n_{01} = \text{id}_{\text{id}_e} + \nu_0 \theta_0$ for some $\nu_0, \nu_1, \nu_2 \in \mathfrak{n}[1]$. The same holds in fact for all terms appearing in (4.30). Our definition of induced concatenation and products \otimes to linear order are therefore sufficient. For example, we compute

$$\begin{aligned} n_{012} \otimes \text{id}_{m_2} &= (\text{id}_{\text{id}_e} + \nu_1 \theta_0 + \nu_2 \theta_2) \otimes (\text{id}_{\text{id}_e} + \alpha \theta_2) \\ &= \text{id}_{\text{id}_e} + \nu_1 \theta_0 + (\nu_2 + \alpha) \theta_2 - \nu_1 \otimes \alpha \theta_2 \\ &= \text{id}_{\text{id}_e} + \nu_1 \theta_0 + (\nu_2 + \alpha - \nu_1 \otimes \alpha) \theta_2. \end{aligned} \quad (4.36)$$

Comparing the coefficient of $\theta_0 \theta_1 \theta_2$ of both sides of equation (4.30), we obtain

$$\gamma = \text{id}_\alpha \otimes \beta - \beta \otimes \text{id}_\alpha + \mu(\alpha, \alpha, \alpha). \quad (4.37)$$

In deriving the latter, we used $\beta \circ (\text{id}_{\mathfrak{s}(\beta)} - \beta) = 0$, which follows immediately from Proposition 2.29. \square

Corollary 4.22. *The induced differentials d_K of $\alpha \in \mathfrak{m}[1]$ and $\beta \in \mathfrak{n}[2]$ are given by*

$$\begin{aligned} d_K \alpha &= -\alpha \otimes \alpha - \mathfrak{s}(\beta), \\ d_K \beta &= -\text{id}_\alpha \otimes \beta + \beta \otimes \text{id}_\alpha - \mu(\alpha, \alpha, \alpha). \end{aligned} \quad (4.38)$$

Proof. This is a direct consequence of the application of the differential d_K to $\{n_{012}\}$ as given in Proposition 4.21. Alternatively, the first of these equations can also be obtained from the application of d_K to $\{m_{01}\}$. \square

Remark 4.23. *From equations (4.38), we can now extract the Chevalley–Eilenberg algebra of an L_∞ -algebra. Assume a basis $(\tau_i), (\sigma_m)$ of $\mathfrak{m} \oplus \mathfrak{n}$ together with the dual basis $(\check{\tau}^i), (\check{\sigma}^m)$ of $\mathfrak{m}^* \oplus \mathfrak{n}^*$. Equations (4.38) should be regarded as the evaluation of*

$$\begin{aligned} d_{\text{CE}} \check{\tau}^i &= -s_m^i \check{\sigma}^m - \frac{1}{2} f_{jk}^i \check{\tau}^j \otimes \check{\tau}^k, \\ d_{\text{CE}} \check{\sigma}^m &= -\frac{1}{2} c_{in}^m (\check{\tau}^i \otimes \check{\sigma}^n - \check{\sigma} \otimes \check{\tau}^i) + \frac{1}{3!} d_{ijk}^m \check{\tau}^i \otimes \check{\tau}^j \otimes \check{\tau}^k, \end{aligned} \quad (4.39)$$

at $\check{\tau}^i = a^i$ and $\check{\sigma}^m = b^m$ with $a = a^i \tau_j$ and $b = b^m \sigma_m$. The constants s_m^i , f_{jk}^i , c_{in}^m and d_{ijk}^m are the generalised structure constants of the 2-term L_∞ -algebra $\mathfrak{n} \xrightarrow{\mu_1} \mathfrak{m}$:

$$\mu_1(\sigma_m) = s_m^i \tau_i, \quad \mu_2(\tau_i, \tau_j) = f_{ij}^k \tau_k, \quad \mu_2(\tau_i, \sigma_m) = c_{im}^n \sigma_n, \quad \mu_3(\tau_i, \tau_j, \tau_k) = d_{ijk}^m \sigma_m. \quad (4.40)$$

The homotopy Jacobi identities follow from the fact that $d_{\text{CE}}^2 = d_K^2 = 0$.

We sum up our findings in the following theorem.

Theorem 4.24. *The functor from a manifold X to descent data for a semistrict principal 2-bundle with structure Lie 2-group $\mathcal{B}\mathcal{G}(\{e\}, M, N)$ on the surjective submersion $X \times \mathbb{R}^{0|1} \rightarrow X$ is parameterised by a 2-term L_∞ -algebra $T_{\text{id}_{\text{id}_e}} N[1] \rightarrow T_{\text{id}_e} M[1]$. The action of the differential d_K on the resulting descent data yields the Chevalley–Eilenberg differential of the 2-term L_∞ -algebra.*

Analogously to Lie groups, we would like to consider an equivalent set of descent data and compare the change of the resulting Chevalley–Eilenberg algebra. This will eventually give us equivalent data $(\tilde{\alpha}, \tilde{\beta}) \in \mathfrak{m}[1] \times \mathfrak{n}[2]$ obtained from $(\alpha, \beta) \in \mathfrak{m}[1] \times \mathfrak{n}[2]$.

Lemma 4.25. *Equivalent descent data $(\{\tilde{m}_{01}\}, \{\tilde{n}_{012}\})$ is related to the previous set of descent data $(\{m_{01}\}, \{n_{012}\})$ by a coboundary $(\{p_0 := p(\theta_0)\}, \{q_{01} := q(\theta_0, \theta_1)\})$ according to*

$$\begin{aligned} q_{01} : \tilde{m}_{01} \otimes p_1 &\Rightarrow p_0 \otimes m_{01} , \\ q_{02} \circ (\tilde{n}_{012} \otimes \text{id}_{p_2}) &= (\text{id}_{p_0} \otimes n_{012}) \circ \mathbf{a}_{p_0, m_{01}, m_{12}} \circ (q_{01} \otimes \text{id}_{m_{12}}) \circ \\ &\quad \circ \mathbf{a}_{\tilde{m}_{01}, p_1, m_{12}}^{-1} \circ (\text{id}_{\tilde{m}_{01}} \otimes q_{12}) \circ \mathbf{a}_{\tilde{m}_{01}, \tilde{m}_{12}, p_2} \end{aligned} \quad (4.41)$$

with

$$p_0 = p - d_K p \theta_0 \quad \text{and} \quad q_{01} = \text{id}_p + \lambda_p(\theta_0 - \theta_1) - \text{id}_{d_K p} \theta_1 - d_K \lambda_p \theta_0 \theta_1 \quad (4.42)$$

for some $p \in N$ and $\lambda_p \in T_p N[1]$.

Proof. The expansion for q_{01} in (4.42) follows from $q_{00} = \text{id}_{p_0}$, cf. Remark 3.27, together with $d_K \text{id}_{d_K p} = 0$. \square

Note that contrary to the previously considered coboundaries, p_0 and q_{01} are points in M near p and in N near id_p , respectively. Our formulæ for linearising the structure maps at p and id_p , however, remain essentially the same, cf. Remark 4.17.

Following Proposition 2.14, we may now combine the coboundaries $(\{m_0\}, \{n_{01}\})$ appearing in (4.30) to a new coboundary $(\{m'_0\}, \{n'_{01}\})$. The diagram

$$\begin{array}{ccc} \begin{array}{ccc} e & \xrightarrow{\text{id}_e} & e \\ m_1 \downarrow & \nearrow n_{01} & \downarrow m_0 \\ e & \xrightarrow{m_{01}} & e \\ p_1 \downarrow & \nearrow q_{01} & \downarrow p_0 \\ e & \xrightarrow{\tilde{m}_{01}} & e \end{array} & = & \begin{array}{ccc} e & \xrightarrow{\text{id}_e} & e \\ m'_1 \downarrow & \nearrow n'_{01} & \downarrow m'_0 \\ e & \xrightarrow{m'_{01}} & e \end{array} \end{array} \quad (4.43)$$

yields the formulæ

$$\begin{aligned}
m'_0 &= p_0 \otimes m_0 , \\
n'_{01} : \tilde{m}_{01} \otimes m'_1 &\Rightarrow m'_0 , \\
n'_{01} &= (\text{id}_{p_0} \otimes n_{01}) \circ \mathbf{a}_{p_0, m_{01}, m_1} \circ (q_{01} \otimes \text{id}_{m_1}) \circ \mathbf{a}_{\tilde{m}_{01}, p_1, m_1}^{-1} .
\end{aligned} \tag{4.44}$$

Hence, \tilde{n}_{012} obeys

$$n'_{02} \circ (\tilde{n}_{012} \otimes \text{id}_{m'_2}) = n'_{01} \circ (\text{id}_{\tilde{m}_{01}} \otimes n'_{12}) \circ \mathbf{a}_{\tilde{m}_{01}, \tilde{m}_{12}, m'_2} . \tag{4.45}$$

Comparing the parameterisation of the coboundary $(\{m_0\}, \{n_{01}\})$ with that of $(\{m'_0\}, \{n'_{01}\})$ is not straightforward as their expansions in the Graßmann-odd coordinates are different. In particular m'_0 and n'_{01} are not the same as $\tilde{m}_0 := \tilde{m}(\theta_0, 0)$ and $\tilde{n}_{01} := \tilde{n}(\theta_0, \theta_1, 0)$, in general. To remedy this, we apply a modification isomorphism $\{o_0 : m'_0 \Rightarrow \tilde{m}_0 \otimes p\}$, taking us from the coboundary $(\{m'_0\}, \{n'_{01}\})$ to the coboundary $(\{\tilde{m}_0\}, \{\tilde{n}_{01}\})$:

$$o_0 \circ n'_{01} = \hat{n}_{01} \circ (\text{id}_{\tilde{m}_{01}} \otimes o_1) \quad \text{with} \quad \{o_0 := o(\theta_0) := q^{-1}(\theta_0, 0)\} , \tag{4.46}$$

where $\hat{n}_{01} : \tilde{m}_{01} \otimes (\tilde{m}_1 \otimes p) \Rightarrow \tilde{m}_0 \otimes p$. It is then easy to see that

$$\tilde{m}(0) = \text{id}_e , \quad \hat{n}_{00} = \text{id}_{\tilde{m}_0 \otimes p} , \quad \text{and} \quad \hat{n}(\theta_0, 0) = \text{id}_{\tilde{m}_0 \otimes p} \tag{4.47}$$

and hence,

$$\hat{n}_{02} \circ (\tilde{n}_{012} \otimes \text{id}_{\tilde{m}_2 \otimes p}) = \hat{n}_{01} \circ (\text{id}_{\tilde{m}_{01}} \otimes \hat{n}_{12}) \circ \mathbf{a}_{\tilde{m}_{01}, \tilde{m}_{12}, \tilde{m}_2 \otimes p} . \tag{4.48}$$

For $\theta_2 = 0$, this equation implies that

$$\tilde{n}_{01} \otimes \text{id}_p = \hat{n}_{01} \circ \mathbf{a}_{\tilde{m}_{01}, \tilde{m}_1, p} . \tag{4.49}$$

Altogether, we have thus constructed coboundary data $(\{\tilde{m}_0\}, \{\tilde{n}_{01}\})$ representing the equivalent descent data $(\{\tilde{m}_{01}, \tilde{n}_{012}\}) \sim (\{m_{01}, n_{012}\})$ according to

$$\tilde{n}_{02} \circ (\tilde{n}_{012} \otimes \text{id}_{\tilde{m}_2}) = \tilde{n}_{01} \circ (\text{id}_{\tilde{m}_{01}} \otimes \tilde{n}_{12}) \circ \mathbf{a}_{\tilde{m}_{01}, \tilde{m}_{12}, \tilde{m}_2} . \tag{4.50}$$

These considerations then lead to the following proposition.

Theorem 4.26. *Let $(\{m_{01}, n_{012}\})$ be any descent data that are parametrised by $(\alpha, \beta) \in \mathfrak{m}[1] \times \mathfrak{n}[2]$. Furthermore, let $(\{\tilde{m}_{01}, \tilde{n}_{012}\})$ be any equivalent descent data that are parametrised by $(\tilde{\alpha}, \tilde{\beta}) \in \mathfrak{m}[1] \times \mathfrak{n}[2]$. Then $\tilde{\alpha}$ and $\tilde{\beta}$ are expressed in terms of α and β according to*

$$\lambda_p : \tilde{\alpha} \otimes p \Rightarrow p \otimes \alpha - d_K p , \tag{4.51a}$$

$$\begin{aligned}
\tilde{\beta} \otimes \text{id}_p &= \mu(\tilde{\alpha}, \tilde{\alpha}, p) + [\text{id}_p \otimes \beta + \mu(p, \alpha, \alpha)] \circ \\
&\quad \circ [-d_K \lambda_p - \lambda_p \otimes \text{id}_\alpha - \mu(\tilde{\alpha}, p, \alpha)] \circ \\
&\quad \circ [-\text{id}_{\mathfrak{s}(d_K \lambda_p)} - \text{id}_{\tilde{\alpha}} \otimes (\lambda_p + \text{id}_{d_K p})] ,
\end{aligned} \tag{4.51b}$$

where $p \in M$ and $\lambda_p \in T_p N[1]$. By construction, equations (4.38) are invariant under this equivalence relation.

Proof: We follow the arguments around (4.41)–(4.50) so that the expansions of $\{m_0\}$, $\{n_{01}\}$, $\{m_{01}\}$, and $\{n_{012}\}$ and $\{\tilde{m}_0\}$, $\{\tilde{n}_{01}\}$, $\{\tilde{m}_{01}\}$, and $\{\tilde{n}_{012}\}$, are those given in Proposition 4.21, with tilded coefficients for tilded quantities. The expansion of the coboundary $(\{p_0\}, \{q_{01}\})$ are given in Lemma 4.25.

Since $q_{01} : \tilde{m}_{01} \otimes p_1 \Rightarrow p_0 \otimes m_{01}$, we find by computing the source and target and using the expansions (see also Proposition 4.21 and Corollary 4.22)

$$\begin{aligned} m_{01} &= \text{id}_e + \alpha(\theta_0 - \theta_1) + [\alpha \otimes \alpha + \mathbf{s}(\beta)]\theta_0\theta_1 = \text{id}_e + \alpha(\theta_0 - \theta_1) - d_K \alpha \theta_0 \theta_1 , \\ \tilde{m}_{01} &= \text{id}_e + \tilde{\alpha}(\theta_0 - \theta_1) + [\tilde{\alpha} \otimes \tilde{\alpha} + \mathbf{s}(\tilde{\beta})]\theta_0\theta_1 = \text{id}_e + \tilde{\alpha}(\theta_0 - \theta_1) - d_K \tilde{\alpha} \theta_0 \theta_1 , \end{aligned} \quad (4.52)$$

that

$$\begin{aligned} \lambda_p : \tilde{\alpha} \otimes p &\Rightarrow p \otimes \alpha - d_K p , \\ d_K \lambda_p : -d_K \tilde{\alpha} \otimes p + \tilde{\alpha} \otimes d_K p &\Rightarrow -d_K p \otimes \alpha - p \otimes d_K \alpha , \end{aligned} \quad (4.53)$$

thus verifying (4.51a).

To compute n'_{01} from (4.44), we need to establish the explicit form of the two associators $\mathbf{a}_{p_0, m_{01}, m_1}$ and $\mathbf{a}_{\tilde{m}_{01}, p_1, m_1}^{-1}$. Both of these become trivial for $\theta_0 = \theta_1$ or $\theta_1 = 0$. We therefore have the following expansions,

$$\begin{aligned} \mathbf{a}_{p_0, m_{01}, m_1} &=: \text{id}_{p_0 \otimes (m_{01} \otimes m_1)} + \mu(p, \alpha, \alpha)\theta_0\theta_1 , \\ \mathbf{a}_{\tilde{m}_{01}, p_1, m_1}^{-1} &=: \text{id}_{(\tilde{m}_{01} \otimes p_1) \otimes m_1} - \mu(\tilde{\alpha}, p, \alpha)\theta_0\theta_1 , \end{aligned} \quad (4.54)$$

defining two maps, which we both denote by μ :

$$\begin{aligned} \mu(p, \alpha, \alpha) : p \otimes (\alpha \otimes \alpha) - (p \otimes \alpha) \otimes \alpha &\Rightarrow 0 , \\ \mu(\tilde{\alpha}, p, \alpha) : \tilde{\alpha} \otimes (p \otimes \alpha) - (\tilde{\alpha} \otimes p) \otimes \alpha &\Rightarrow 0 . \end{aligned} \quad (4.55)$$

Upon substituting these expressions together with those for $\{p_{01}\}$, $\{q_0\}$ and $\{n_{01}\}$, $\{m_1\}$ into (4.44), we find

$$\begin{aligned} n'_{01} &= \text{id}_p + (\theta_0 - \theta_1)\lambda_p + \text{id}_{p \otimes \alpha - d_K p} \theta_1 + \\ &\quad + [\text{id}_p \otimes \beta + \mu(p, \alpha, \alpha)] \circ [-d_K \lambda_p - \lambda_p \otimes \text{id}_\alpha - \mu(\tilde{\alpha}, p, \alpha)]\theta_0\theta_1 . \end{aligned} \quad (4.56)$$

Here, we relied on the fact that each of the terms in (4.44) can be written as $\text{id}_p + \theta_0\pi_1 + \theta_1\pi_2$, where $\pi_{1,2} \in T_p N[1]$, and for these, the linearised concatenation is well-defined.

Finally, we perform the modification transformation $o_0 : m'_0 \Rightarrow \tilde{m}_0 \otimes p$ with $\{o_0^{-1} = q(\theta_0, 0)\}$ which we have introduced in (4.46),

$$o_0 \circ n'_{01} = \hat{n}_{01} \circ (\text{id}_{\tilde{m}_{01}} \otimes o_1) \iff o_0^{-1} \circ \hat{n}_{01} = n'_{01} \circ (\text{id}_{\tilde{m}_{01}} \otimes o_1^{-1}) , \quad (4.57)$$

to obtain $\hat{n}_{01} : \tilde{m}_{01} \otimes (\tilde{m}_1 \otimes p) \Rightarrow \tilde{m}_0 \otimes p$. Using (4.49) and $\{o_0^{-1} = q(\theta_0, 0)\}$, this can be rewritten as

$$q(\theta_0, 0) \circ (\tilde{n}_{01} \otimes \text{id}_p) \circ \mathbf{a}_{\tilde{m}_{01}, \tilde{m}_1, p}^{-1} = n'_{01} \circ [\text{id}_{\tilde{m}_{01}} \otimes q(\theta_1, 0)] . \quad (4.58)$$

To evaluate this expression we need to fix the expansion of the associator, which we do according to

$$\mathbf{a}_{\tilde{m}_{01}, \tilde{m}_1, p}^{-1} =: \text{id}_{(\tilde{m}_{01} \otimes \tilde{m}_1) \otimes p} - \mu(\tilde{\alpha}, \tilde{\alpha}, p) \theta_0 \theta_1 , \quad (4.59)$$

where $\mu(\tilde{\alpha}, \tilde{\alpha}, p) : \tilde{\alpha} \otimes (\tilde{\alpha} \otimes p) - (\tilde{\alpha} \otimes \tilde{\alpha}) \otimes p \Rightarrow 0$. Substituting this expression, (4.42), and (4.56) into (4.58), we find after some algebraic manipulations $\tilde{n}_{01} = \text{id}_e + \text{id}_{\tilde{\alpha}} \theta_0 + \tilde{\beta} \theta_0 \theta_1$ with

$$\begin{aligned} \tilde{\beta} \otimes \text{id}_p &= \mu(\tilde{\alpha}, \tilde{\alpha}, p) + [\text{id}_p \otimes \beta + \mu(p, \alpha, \alpha)] \circ \\ &\circ [-\text{d}_K \lambda_p - \lambda_p \otimes \text{id}_\alpha - \mu(\tilde{\alpha}, p, \alpha)] \circ [-\text{id}_{\mathfrak{s}(\text{d}_K \lambda_p)} - \text{id}_{\tilde{\alpha}} \otimes (\lambda_p + \text{id}_{\text{d}_K p})] \end{aligned} \quad (4.60)$$

verifying (4.51b). Note that $\mathfrak{t}(\tilde{\beta}) = 0$ as required. This concludes the proof. \square

Finally, we would like to emphasise that given $\lambda_p \in T_p N[1]$, we can always construct a $\lambda \in \mathfrak{n}[1]$ and vice versa.

Definition 4.27. Let $p \in M$ and $\lambda_p \in T_p N[1]$ be given as in Proposition 4.26. We define a 2-cell $\lambda \in \mathfrak{n}[1]$ by setting

$$\lambda := (\lambda_p \otimes \text{id}_{\bar{p}}) \circ \mathbf{a}_{\tilde{\alpha}, p, \bar{p}}^{-1} , \quad (4.61)$$

that is, $\lambda : \tilde{\alpha} \Rightarrow (p \otimes \alpha) \otimes \bar{p} - \text{d}_K p \otimes \bar{p}$, where $\bar{p} \in M$ with $p \otimes \bar{p} = \text{id}_e = \bar{p} \otimes p$ and $\mathbf{a}_{\tilde{\alpha}, p, \bar{p}} : (\tilde{\alpha} \otimes p) \otimes \bar{p} \Rightarrow \tilde{\alpha} \otimes (p \otimes \bar{p})$.

Due to the naturalness of the associator, it is straightforward to see that λ_p can be expressed in terms of λ by means of

$$\lambda_p = \mathbf{a}_{p \otimes \alpha + \text{d}_K p, \bar{p}, p} \circ [(\lambda \circ \mu_{\tilde{\alpha}, p, \bar{p}}) \otimes \text{id}_p] \circ \mathbf{a}_{\tilde{\alpha} \otimes p, \bar{p}, p}^{-1} . \quad (4.62)$$

4.4. Example: Strict Lie 2-groups

As a consistency check, let us now consider a class of examples. Since it is notoriously difficult to construct non-trivial examples of Lie 2-groups which are not strict, we have to consider the strict case. That is, we start from descent data for strict principal 2-bundles in the general Lie 2-group framework. For such bundles, we have $n_{012} = \ell_{02}^{-1} \circ (\ell_{01} \otimes \ell_{12})$. One can check that then

$$\ell_{01} = n_{01} \otimes \text{id}_{\bar{m}_1} = \text{id}_{\text{id}_e} + \text{id}_\alpha (\theta_0 - \theta_1) + (\beta + \text{id}_{\alpha \otimes \alpha}) \theta_0 \theta_1 , \quad (4.63)$$

which yields the following.

Lemma 4.28. *For strict Lie 2-groups, the functor of descent data reads as*

$$\begin{aligned} m_{01} &= \text{id}_e + \alpha(\theta_0 - \theta_1) + \theta_0\theta_1[\alpha \otimes \alpha + \mathfrak{s}(\beta)] , \\ n_{012} &= \text{id}_{\text{id}_e} + \text{id}_\alpha(\theta_0 - \theta_2) + \beta(\theta_0\theta_1 + \theta_1\theta_2 - \theta_0\theta_2) + \\ &\quad + \text{id}_{\alpha \otimes \alpha + \mathfrak{s}(\beta)}\theta_0\theta_2 + (\text{id}_\alpha \otimes \beta - \beta \otimes \text{id}_\alpha)\theta_0\theta_1\theta_2 , \end{aligned} \quad (4.64)$$

which implies

$$\text{d}_K\alpha = -\alpha \otimes \alpha - \mathfrak{s}(\beta) \quad \text{and} \quad \text{d}_K\beta = -\text{id}_\alpha \otimes \beta + \beta \otimes \text{id}_\alpha . \quad (4.65)$$

To compare with the literature, we need to translate these results into expressions using crossed modules of Lie groups.

Proposition 4.29. *In terms of crossed modules of Lie groups $(\mathbf{H} \xrightarrow{\partial} \mathbf{G}, \triangleright)$, the functor of descent data is given by Čech 1- and 2-cochains $\{g_{01}\}$ and $\{h_{012}\}$ with values in the Lie groups \mathbf{G} and \mathbf{H} , respectively. These are parameterised by $a \in \mathfrak{g}[1]$ and $b \in \mathfrak{h}[2]$, where \mathfrak{g} and \mathfrak{h} are the Lie algebras of \mathbf{G} and \mathbf{H} , according to*

$$g_{01} = \mathbb{1}_G + a(\theta_0 - \theta_1) + \left\{ \frac{1}{2}[a, a] - \partial(b) \right\} \theta_0\theta_1 \quad (4.66a)$$

and

$$h_{012} = \mathbb{1}_H + b(\theta_0\theta_1 + \theta_1\theta_2 - \theta_0\theta_2) + (a \triangleright b)\theta_0\theta_1\theta_2 . \quad (4.66b)$$

The action of the differential d_K translates to

$$\text{d}_K a = -\frac{1}{2}[a, a] + \partial(b) \quad \text{and} \quad \text{d}_K b = -a \triangleright b . \quad (4.67)$$

Proof. Starting from (4.64) and (4.65), we follow Proposition 2.36 and assume $\mathbf{B}\mathcal{G} = (M, N) = (\mathbf{G}, \mathbf{H} \rtimes \mathbf{G})$. We then identify $\alpha = a \in \mathfrak{g}[1]$ and, since $\mathfrak{t}(\beta) = 0$, we set $\beta = (b, 0)$ with $b \in \mathfrak{h}[2]$. Moreover, $m_{01} = g_{01}$ and $n_{012} = (h_{012}, g_{012}) = (h_{012}, g_{01}g_{12})$. \square

These are the expressions that were already obtained in Jurčo [33].

Furthermore, combining the results of Proposition 4.26 and Definition 4.27 with the interchange law (2.5), we arrive after a few algebraic manipulations at

$$\begin{aligned} \lambda : \tilde{\alpha} &\Rightarrow p \otimes \alpha \otimes \bar{p} - \text{d}_K p \otimes \bar{p} , \\ \tilde{\beta} &= [\text{id}_p \otimes \beta \otimes \text{id}_{\bar{p}}] \circ [-\text{d}_K \lambda - \lambda \otimes \lambda] . \end{aligned} \quad (4.68)$$

Translated into crossed modules of Lie groups, this takes the following form.

Proposition 4.30. *Equivalent descent data $(\{\tilde{g}_{01}\}, \{\tilde{h}_{012}\})$ is parameterised by $\tilde{a} \in \mathfrak{g}[1]$ and $\tilde{b} \in \mathfrak{h}[2]$, which is related to the parameters $a \in \mathfrak{g}[1]$ and $b \in \mathfrak{h}[2]$ of $(\{g_{01}\}, \{h_{012}\})$ by the following equations.*

$$\tilde{a} = pap^{-1} + p d_K p^{-1} - \partial(\lambda) , \quad (4.69a)$$

$$\tilde{b} = p \triangleright b - d_K \lambda - \tilde{a} \triangleright \lambda - \frac{1}{2}[\lambda, \lambda] . \quad (4.69b)$$

Proof. Setting $\lambda =: (\lambda^{\mathfrak{h}}, \lambda^{\mathfrak{g}})$ with $\lambda^{\mathfrak{g}} = pap^{-1} + p d_K p^{-1}$ in (4.68), we obtain (4.69a) by considering the images of the source and target maps and renaming $\lambda^{\mathfrak{h}}$ by λ . In addition, since $\text{id}_p = (\mathbb{1}_H, p)$, the term $\text{id}_p \otimes \beta \otimes \text{id}_{\bar{p}}$ can be rewritten as $\text{id}_p \otimes \beta \otimes \text{id}_{\bar{p}} = (p \triangleright b, 0)$. Likewise, the expression $-d_K \lambda - \lambda \otimes \lambda$ becomes $-d_K \lambda - \lambda \otimes \lambda = (-d_K \lambda^{\mathfrak{h}} - \lambda^{\mathfrak{g}} \triangleright \lambda^{\mathfrak{h}}, -d_K \lambda^{\mathfrak{g}} - \frac{1}{2}[\lambda^{\mathfrak{g}}, \lambda^{\mathfrak{g}}])$. Thus,

$$[\text{id}_p \otimes \beta \otimes \text{id}_{\bar{p}}] \circ [-d_K \lambda - \lambda \otimes \lambda] = (p \triangleright b - d_K \lambda^{\mathfrak{h}} - \tilde{a} \triangleright \lambda^{\mathfrak{h}} - \frac{1}{2}[\lambda^{\mathfrak{h}}, \lambda^{\mathfrak{h}}], 0) , \quad (4.70)$$

where we have used the Peiffer identity (2.28). With $\tilde{\beta} = (\tilde{b}, 0)$, we therefore find from the above expression the equation (4.69b) after renaming $\lambda^{\mathfrak{h}}$ by λ . \square

4.5. Comment on differentiation and categorical equivalence

Recall from Proposition 2.25 that every weak 2-group is categorically equivalent to a special weak 2-group given in terms of a group G , an Abelian group H , a representation α of G on H and an element $[a] \in H^3(G, H)$. The corresponding Proposition for Lie 2-algebras 2.43 states that semistrict Lie 2-algebras are equivalent to special Lie 2-algebras given in terms of a Lie algebra \mathfrak{g} , a representation α of \mathfrak{g} on a vector space V and an element $[J] \in H^3(\mathfrak{g}, V)$.

It is now tempting to assume that the natural integration process factors through categorical equivalence and therefore special Lie 2-algebras can be integrated to special Lie 2-groups. However, Baez & Lauda proved a no-go theorem [4, Section 8.5], which shows that certain special Lie 2-algebras can be integrated to 2-groups, which, however, can be turned into topological 2-groups only for the strict case $a = 0$. In particular, consider the case of a special Lie 2-algebra with $V = \mathfrak{u}(1)$. We have $H^3(\mathfrak{g}, \mathfrak{u}(1)) \cong \mathbb{R}$. The latter contains a lattice $\cong \mathbb{Z}$, which can be embedded into $H^3(G, \text{U}(1))$, yielding the integration to a 2-group. In the topological case, however, we have to use continuous cohomology, for which $H_{\text{cont.}}^3(G, \text{U}(1)) = 0$.

The differentiation of Lie 2-groups we performed in this section is the inverse operation to this integration. As integration does not commute with categorical equivalence, neither will differentiation.

5. Semistrict higher gauge theory

We now put the results of the previous section together to a description of semistrict principal 2-bundles with connective structure. We first discuss the local case¹⁴, which can be readily derived from the Maurer–Cartan equations of an L_∞ -algebra. We then give the global description in terms of non-Abelian Deligne cohomology sets.

5.1. Local semistrict higher gauge theory

Let X be a smooth manifold and Ω_X^p be the sheaf of differential p -forms on X . Furthermore, let $U \subseteq X$ be an open subset. Local semistrict higher gauge theory corresponds to the Maurer–Cartan equation (A.7) for a degree 1-element of the L_∞ -algebra arising from the tensor product of $\Omega_X^\bullet = \bigoplus_p \Omega_X^p$ and a gauge L_∞ -algebra L . The corresponding infinitesimal gauge transformations are the gauge transformations of the Maurer–Cartan equations (A.8). To make this explicit, we wish to recall the following proposition.

Proposition 5.1. *A tensor product of a differential graded algebra \mathfrak{a} and an L_∞ -algebra L comes with a natural L_∞ -structure. The grading of an element of $\mathfrak{a} \otimes L$ is the sum of its individual gradings. Moreover, for a tuple of elements $(a_1 \otimes \ell_1, \dots, a_i \otimes \ell_i)$ of $\mathfrak{a} \otimes L$, the higher products $\tilde{\mu}_i$ read as*

$$\tilde{\mu}_i(a_1 \otimes \ell_1, \dots, a_i \otimes \ell_i) = \begin{cases} (da_1) \otimes \ell_1 + (-1)^{\deg(a_1)} a_1 \otimes \mu_1(\ell_1) & \text{for } i = 1, \\ \chi(a_1 a_2 \dots a_i \otimes \mu_i(\ell_1, \dots, \ell_i)) & \text{for } i > 1. \end{cases} \quad (5.1)$$

Here, μ_i are the higher products in L , \deg denotes the degrees in \mathfrak{a} and $\chi = \pm 1$ is the Koszul sign arising from moving graded elements of \mathfrak{a} past graded elements of L .

Proof. The homotopy Jacobi identities, given in (A.2), for the higher products $\tilde{\mu}_i$ are readily checked. \square

As an example, let us work out the details for the case where \mathfrak{a} is the de Rham complex on an open subset U of a manifold X and L is a 2-term L_∞ -algebra.

Example 5.2. *The tensor product of Ω_X^\bullet and the 2-term L_∞ -algebra $L = L_{-1} \oplus L_0 = \mathfrak{n} \oplus \mathfrak{m}$ consists of the following graded subspaces*

$$\begin{aligned} & \underbrace{H^0(U, \Omega_X^0 \otimes \mathfrak{n})}_{\text{degree } -1} \oplus \underbrace{H^0(U, \Omega_X^0 \otimes \mathfrak{m}) \oplus H^0(U, \Omega_X^1 \otimes \mathfrak{n})}_{\text{degree } 0} \oplus \\ & \qquad \oplus \underbrace{H^0(U, \Omega_X^1 \otimes \mathfrak{m}) \oplus H^0(U, \Omega_X^2 \otimes \mathfrak{n})}_{\text{degree } 1} \oplus \dots \end{aligned} \quad (5.2)$$

¹⁴For more details on the local case, see Sati, Schreiber & Stasheff [35].

For an element ϕ of degree 1, the homotopy Maurer–Cartan equations (A.7) read as

$$-\tilde{\mu}_1(\phi) - \frac{1}{2}\tilde{\mu}_2(\phi, \phi) + \frac{1}{3!}\tilde{\mu}_3(\phi, \phi, \phi) = 0, \quad (5.3)$$

which are invariant under the transformations

$$\delta\phi = \tilde{\mu}_1(\lambda) - \tilde{\mu}_2(\lambda, \phi) - \frac{1}{2}\tilde{\mu}_3(\lambda, \phi, \phi). \quad (5.4)$$

Proposition 5.3. *The homotopy Maurer–Cartan equations (5.3) and the transformations (5.4) are equivalent to the equations*

$$\begin{aligned} \mathcal{F} &:= dA + \frac{1}{2}\mu_2(A, A) - \mu_1(B) = 0, \\ H &:= dB + \mu_2(A, B) - \frac{1}{3!}\mu_3(A, A, A) = 0, \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} \delta A &= d\alpha + \mu_2(A, \alpha) + \mu_1(\beta), \\ \delta B &= d\beta + \mu_2(A, \beta) + \mu_2(B, \alpha) + \frac{1}{2}\mu_3(\alpha, A, A), \end{aligned} \quad (5.6)$$

where $A \in H^0(U, \Omega_X^1 \otimes \mathfrak{m})$, $B \in H^0(U, \Omega_X^2 \otimes \mathfrak{n})$, $\alpha \in H^0(U, \Omega_X^0 \otimes \mathfrak{m})$, and $\beta \in H^0(U, \Omega_X^1 \otimes \mathfrak{n})$.

Proof. This trivially follows by identifying $\phi = A - B$ and $\lambda = \alpha - \beta$ in (5.3) and (5.4). \square

Remark 5.4. *For trivial μ_3 , equations (5.5) reduce to the field equations for a flat connective structure of a principal 2-bundle with strict structure 2-group. Equations (5.6) describe infinitesimal gauge transformations acting on this connective structure.*

Note that there are equivalence relations between gauge transformations, which have the same effect on A and B . These are given by

$$\delta\alpha = \mu_1(\gamma) \quad \text{and} \quad \delta\beta = -d\gamma - \mu_2(A, \gamma), \quad (5.7)$$

where $\gamma \in H^0(U, \Omega_X^0 \otimes \mathfrak{n})$.

Finally, we would like to stress that the local equations for semistrict higher gauge theory on principal n -bundles with connective structure is similarly derived by taking tensor products of Ω_X^\bullet with n -term L_∞ -algebras.

5.2. Finite gauge transformations of the local connective structure

Having derived curvature and infinitesimal gauge transformation for semistrict higher gauge theory, let us now turn to the finite gauge transformations. Here, we rely on the results of section 4. We shall first discuss ordinary gauge theory as an example, which serves as a guide in the discussion of the corresponding structure for higher gauge theory.

In the following, let X be a manifold, U an open subset of X , and G a Lie group. Recall the induced differential d_K on $\mathfrak{g}[n]$ as defined in Lemma 4.8, which defines a complex

$$\mathfrak{g}[1] \xrightarrow{d_K} \mathfrak{g}[2] \xrightarrow{d_K} \cdots . \quad (5.8)$$

We now consider the tensor product of this complex by Ω_X^\bullet , which is similar to the tensor product of a 2-term L_∞ -algebra and Ω_X^\bullet discussed in Example 5.2. We trivially have the following lemma:

Lemma 5.5. *The induced differential d_K can be extended to a differential $d_c := d - d_K$, where d denotes the de Rham differential. The latter acts on elements $\omega \otimes a$ of $\Omega_X^p \otimes \mathfrak{g}[k]$ according to*

$$d_c(\omega \otimes a) := d\omega \otimes a - (-1)^p \omega \otimes d_K a . \quad (5.9)$$

It satisfies $d_c \circ d_c = 0$ and defines a complex

$$\begin{aligned} \Omega_X^0 \otimes \mathfrak{g}[1] &\xrightarrow{d_c} \Omega_X^1 \otimes \mathfrak{g}[1] \oplus \Omega_X^0 \otimes \mathfrak{g}[2] \xrightarrow{d_c} \\ &\xrightarrow{d_c} \Omega_X^2 \otimes \mathfrak{g}[1] \oplus \Omega_X^1 \otimes \mathfrak{g}[2] \oplus \Omega_X^0 \otimes \mathfrak{g}[3] \xrightarrow{d_c} \cdots , \end{aligned} \quad (5.10)$$

where the total grading is the sum of the de Rham grading and the Graßmann grading.

We can now consider a Lie-algebra valued connection 1-form A on U and its associated curvature. Because of

$$d_c A = dA - d_K A = dA + \frac{1}{2}[A, A] = F , \quad (5.11)$$

we conclude the following.

Proposition 5.6. *When acting on elements A of $H^0(U, \Omega_X^1 \otimes \mathfrak{g})$, d_c defines the curvature of the local connection 1-form A .*

In Proposition 4.10, we showed that the equation $d_K a + \frac{1}{2}[a, a] = 0$ was invariant under $a \mapsto \tilde{a} = pap^{-1} + pd_K p^{-1}$. Since d_K and d have the same algebraic properties, we derived the well-known statement

Proposition 5.7. *If a local connection 1-form A is flat, its curvature $F := d_c A = 0$ is invariant under the transformation*

$$A \mapsto \tilde{A} = pAp^{-1} + pdp^{-1} \quad (5.12)$$

for any $p \in H^0(U, \mathcal{G}_X)$, where \mathcal{G}_X is the sheaf of smooth G -valued functions on X . Such transformations are called gauge transformations.

Note also the following consequence.

Corollary 5.8. *At infinitesimal level, the transformations (5.12) amount to*

$$A \mapsto \tilde{A} = d\pi + [A, \pi] , \quad (5.13)$$

where $\pi \in H^0(U, \Omega_X^0 \otimes \mathfrak{g})$. They match the gauge transformations in Proposition 5.3 for the 2-term L_∞ -algebra $\{0\} \rightarrow \mathfrak{g}$.

We now come to the case of a local connective structure on semistrict principal 2-bundles. Here, we combine the differential d_K acting on the double complex

$$\begin{array}{ccccccc} \mathfrak{n}[1] & \xrightarrow{d_K} & \mathfrak{n}[2] & \xrightarrow{d_K} & \mathfrak{n}[3] & \xrightarrow{d_K} & \dots \\ \downarrow s & & \downarrow s & & \downarrow s & & \\ \mathfrak{m}[1] & \xrightarrow{d_K} & \mathfrak{m}[2] & \xrightarrow{d_K} & \mathfrak{m}[3] & \xrightarrow{d_K} & \dots \end{array} \quad (5.14)$$

with the de Rham differential to a new differential $d_c = d - d_K$.

Definition 5.9. *When acting on an element (A, B) of $H^0(U, \Omega_X^1 \otimes \mathfrak{m}) \times H^0(U, \Omega_X^2 \otimes \mathfrak{n})$, d_c defines the curvature forms of the connective structure (A, B) :*

$$\begin{aligned} \mathcal{F} &:= d_c A = dA + A \otimes A + \mathfrak{s}(B) , \\ H &:= d_c B = dB + \text{id}_A \otimes B - B \otimes \text{id}_A + \mu(A, A, A) . \end{aligned} \quad (5.15)$$

Proposition 5.10. *The curvature forms defined in equations (5.15) can be identified with the curvature forms (5.5) arising from the 2-term L_∞ -algebra corresponding to the semistrict 2-group $\mathbf{BG} = (\{e\}, M, N)$.*

Proof. First, recall the identification of the linearised tensor products with the higher products of the L_∞ -algebra from Remark 4.23. For all $\alpha_{1,2} \in \mathfrak{m}[1]$ and $\beta \in \mathfrak{n}[1]$, we identify

$$\begin{aligned} \alpha_1 \otimes \alpha_2 + \alpha_2 \otimes \alpha_1 &= \mu_2(\alpha_1, \alpha_2) , \\ \text{id}_{\alpha_1} \otimes \beta + \beta \otimes \text{id}_{\alpha_1} &= \mu_2(\alpha_1, \beta) , \\ \mu(\alpha_1, \alpha_1, \alpha_2) - \mu(\alpha_1, \alpha_2, \alpha_1) + \mu(\alpha_2, \alpha_1, \alpha_1) &= \frac{1}{2}\mu_3(\alpha, \alpha, \alpha) . \end{aligned} \quad (5.16)$$

Together with $\mathfrak{s}(\beta) = \mu_1(\beta)$ and inversion of the signs of μ_1 and μ_3 as discussed in Remark 2.40, equations (5.15) turn into

$$\begin{aligned} \mathcal{F} &= dA + \frac{1}{2}\mu_2(A, A) - \mu_1(B) , \\ H &= dB + \mu_2(A, B) - \frac{1}{3!}\mu_3(A, A, A) . \end{aligned} \quad (5.17)$$

□

In Proposition 4.26, we showed that the equations $d_K \alpha = -\alpha \otimes \alpha - s(\beta)$ and $d_K \beta = -\text{id}_\alpha \otimes \beta + \beta \otimes \text{id}_\alpha - \mu(\alpha, \alpha, \alpha)$ were invariant under (4.51a) and (4.51b). Again, since d_K and d have the same algebraic properties, we have derived the following statement.

Proposition 5.11. *If the curvatures \mathcal{F} and H corresponding to the connective structure defined by A and B vanish, then they are invariant under the transformation*

$$\Lambda_p : \tilde{A} \otimes p \Rightarrow p \otimes A - dp, \quad (5.18a)$$

$$\begin{aligned} \tilde{B} \otimes \text{id}_p &= \mu(\tilde{A}, \tilde{A}, p) + [\text{id}_p \otimes B + \mu(p, A, A)] \circ \\ &\quad \circ [-d\Lambda_p - \Lambda_p \otimes \text{id}_A - \mu(\tilde{A}, p, A)] \circ \\ &\quad \circ [-\text{id}_{s(d\Lambda_p)} - \text{id}_{\tilde{A}} \otimes (\Lambda_p + \text{id}_{dp})], \end{aligned} \quad (5.18b)$$

where $p \in H^0(U, \mathcal{G}_X)$ and $\Lambda_p \in H^0(U, \Omega^1(T_p N[1]))$. We shall refer to such transformations as gauge transformations.

Again, note also the following fact.

Proposition 5.12. *At the infinitesimal level, the gauge transformations (5.18) become*

$$\begin{aligned} \delta A &= d\alpha + \mu_2(A, \alpha) + \mu_1(\beta), \\ \delta B &= d\beta + \mu_2(A, \beta) + \mu_2(B, \alpha) + \frac{1}{2}\mu_3(\alpha, A, A), \end{aligned} \quad (5.19)$$

where $\alpha \in H^0(U, \Omega_X^0 \otimes \mathfrak{m})$ and $\beta \in H^0(U, \Omega_X^1 \otimes \mathfrak{n})$. Hence, they agree with the gauge transformations in Proposition 5.3 for the 2-term L_∞ -algebra $L = \mathfrak{n} \oplus \mathfrak{m}$ situated in degrees -1 and 0.

Proof. Linearising (5.18b) in the gauge parameters such that $p = \text{id}_e - \alpha$ and $\Lambda = \text{id}_{\text{id}_e} - \beta$, we obtain

$$\begin{aligned} (B + \delta B) \otimes (\text{id}_{\text{id}_e} - \text{id}_\alpha) &= -\mu(A, A, \alpha) + \text{id}_\alpha \otimes B - \mu(\alpha, A, A) + d\beta - \beta \otimes \text{id}_A + \\ &\quad + \mu_3(A, \alpha, A) + \text{id}_{s(d\beta)} - \text{id}_A \otimes (-\beta - \text{id}_{d\alpha}). \end{aligned} \quad (5.20)$$

We now use $\text{id}_{s(d\beta)} + \text{id}_A \otimes \text{id}_{d\alpha} = 0$ as well as the identification of the linearised tensor products of the semistrict Lie 2-group $\mathbf{B}\mathcal{G}(\{e\}, M, N)$ with the higher products of the corresponding Lie 2-algebra as done in equations (5.16). This yields the infinitesimal transformation for B as given in (5.19). Analogously, we linearise the map (5.18a) according to

$$\delta A - A \otimes \alpha + s(\beta) = -\alpha \otimes A + d\alpha, \quad (5.21)$$

which reproduces the infinitesimal transformation of A as given in (5.19). \square

5.3. Connective structure

In the following, let X be a smooth manifold with cover $\mathfrak{U} = \{U_a\}$. Consider a (normalised) semistrict principal 2-bundle with semistrict structure 2-group $\mathbf{B}\mathcal{G} = (\{e\}, M, N)$. It is characterised by the transition functions $(\{m_{ab}\}, \{n_{abc}\})$. Next, we would like to equip the bundle with a connective structure:

Definition 5.13. A connective structure on a semistrict principal 2-bundle with semistrict structure 2-group $\mathbf{B}\mathcal{G} = (\{e\}, M, N)$ consists of $(\{A_a\}, \{B_a\}, \{\Lambda_{ab}\})$, where $A_a \in H^0(U_a, \Omega_X^1 \otimes \mathfrak{m})$, $B_a \in H^0(U_a, \Omega_X^2 \otimes \mathfrak{n})$, and $\Lambda_{ab} \in H^0(U_a \cap U_b, \Omega_X^1 \otimes T_{m_{ab}}N[1])$.

It is clear from Proposition 5.11, how $(\{A_a\}, \{B_a\})$ are patched together on non-empty overlaps $U_a \cap U_b$.

Corollary 5.14. Let $(\{A_a\}, \{B_a\}, \{\Lambda_{ab}\})$ be a connective structure on a semistrict principal 2-bundle. The cocycle conditions for the differential forms $(\{A_a\}, \{B_a\})$ on non-empty overlaps $U_a \cap U_b$ are as follows:

$$\Lambda_{ab} : A_b \otimes m_{ab} \Rightarrow m_{ab} \otimes A_a - \mathrm{d}m_{ab} , \quad (5.22a)$$

$$\begin{aligned} B_b \otimes \mathrm{id}_{m_{ab}} &= \mu(A_b, A_b, m_{ab}) + [\mathrm{id}_{m_{ab}} \otimes B_a + \mu(m_{ab}, A_a, A_a)] \circ \\ &\quad \circ [-\mathrm{d}\Lambda_{ab} - \Lambda_{ab} \otimes \mathrm{id}_A - \mu(A_b, m_{ab}, A_a)] \circ \\ &\quad \circ [-\mathrm{id}_{\mathfrak{s}(\mathrm{d}\Lambda_{ab})} - \mathrm{id}_{A_b} \otimes (\Lambda_{ab} + \mathrm{id}_{\mathrm{d}m_{ab}})] . \end{aligned} \quad (5.22b)$$

Note that the condition that two transformations of the form (5.22a) combine to a third one on non-empty triple overlaps of patches yields the cocycle condition for $\{\Lambda_{ab}\}$. To derive this condition, let us consider

$$\begin{aligned} \Lambda_{ab} : A_b \otimes m_{ba} &\Rightarrow m_{ba} \otimes A_a - \mathrm{d}m_{ba} , \\ \Lambda_{bc} : A_c \otimes m_{cb} &\Rightarrow m_{cb} \otimes A_b - \mathrm{d}m_{cb} , \\ \Lambda_{ac} : A_c \otimes m_{ca} &\Rightarrow m_{ca} \otimes A_a - \mathrm{d}m_{ca} , \end{aligned} \quad (5.23)$$

over a non-empty triple overlap $U_a \cap U_b \cap U_c$. Recall also that

$$n_{abc} : m_{ab} \otimes m_{bc} \Rightarrow m_{ac} . \quad (5.24)$$

Chasing the commutative diagram relating two possible ways of going from $(A_b \otimes m_{ba}) \otimes m_{ac}$ to $m_{bc} \otimes A_c - \mathrm{d}m_{bc}$, we obtain the following.

Proposition 5.15. *The Čech 1-cochain $\{\Lambda_{ab}\}$, gluing together the local connective structures, has to satisfy the following condition:*

$$\begin{aligned} \Lambda_{cb} \circ (\text{id}_{A_b} \otimes n_{bac}) \circ \mathbf{a}_{A_b, m_{ba}, m_{ac}} &= \\ &= (n_{bac} \otimes \text{id}_{A_c} - \text{d}n_{bac}) \circ (\mathbf{a}_{m_{ba}, m_{ac}, A_c}^{-1} - \text{id}_{\text{d}(m_{ba} \otimes m_{ac})}) \circ \\ &\quad \circ (\text{id}_{m_{ba}} \otimes \Lambda_{ca} - \text{id}_{\text{d}m_{ba} \otimes m_{ac}}) \circ (\mathbf{a}_{m_{ba}, A_c, m_{ac}} - \text{id}_{\text{d}m_{ba} \otimes m_{ac}}) \circ (\Lambda_{ab} \otimes \text{id}_{m_{ac}}) . \end{aligned} \quad (5.25)$$

In the above equation, we used an intuitive notation: for instance, $n_{bac} \otimes \text{id}_{A_c} - \text{d}n_{bac}$ has to be understood as

$$n_{bac} \otimes \text{id}_{A_c} - \text{d}n_{bac} : (m_{ba} \otimes m_{ac}) \otimes A_c - \text{d}(m_{ba} \otimes m_{ac}) \Rightarrow m_{bc} \otimes A_c - \text{d}m_{bc} . \quad (5.26)$$

Finally, we would like to describe the action of a coboundary on the cocycle $\{\Lambda_{ab}\}$. Specifically, this means that we compare the following expressions:

$$\begin{aligned} \Lambda_{ab} : A_b \otimes m_{ba} &\Rightarrow m_{ba} \otimes A_a - \text{d}m_{ba} , \\ \tilde{\Lambda}_{ab} : \tilde{A}_b \otimes \tilde{m}_{ba} &\Rightarrow \tilde{m}_{ba} \otimes \tilde{A}_a - \text{d}\tilde{m}_{ba} , \end{aligned} \quad (5.27)$$

where the gauge transformations are given locally by

$$\begin{aligned} \Lambda_a : \tilde{A}_a \otimes m_a &\Rightarrow m_a \otimes A_a - \text{d}m_a , \\ n_{ab} : \tilde{m}_{ab} \otimes m_b &\Rightarrow m_a \otimes m_{ab} . \end{aligned} \quad (5.28)$$

Again, chasing the corresponding commutative diagram relating two possible ways of going from $(\tilde{A}_a \otimes m_a) \otimes m_{ab}$ to $(\tilde{m}_{ab} \otimes \tilde{A}_b) \otimes m_b - \text{d}\tilde{m}_{ab} \otimes m_b$ yields the following.

Proposition 5.16. *Two cocycles $\{\Lambda_{ab}\}$ and $\{\tilde{\Lambda}_{ab}\}$ are equivalent if and only if*

$$\begin{aligned} &(\tilde{\Lambda}_{ba} \otimes \text{id}_{m_b}) \circ \mathbf{a}_{\tilde{A}_a, \tilde{m}_{ab}, m_b}^{-1} \circ (\text{id}_{\tilde{A}_a} \otimes n_{ab}) \circ \mathbf{a}_{\tilde{A}_a, m_a, m_{ab}} = \\ &= (\mathbf{a}_{\tilde{m}_{ab}, \tilde{A}_b, m_b}^{-1} - \text{id}_{\text{d}\tilde{m}_{ab} \otimes m_b}) \circ (\text{id}_{\tilde{m}_{ab}} \otimes \Lambda_b^{-1} - \text{id}_{\text{d}\tilde{m}_{ab} \otimes m_b}) \circ (\mathbf{a}_{m_{ab}, m_b, A_b} - \text{id}_{\text{d}(m_{ab} \otimes m_b)}) \circ \\ &\quad \circ (n_{ab}^{-1} \otimes \text{id}_{A_b} - \text{d}n_{ab}^{-1}) \circ (\mathbf{a}_{m_a, m_{ab}, A_b}^{-1} - \text{id}_{\text{d}(m_a \otimes m_{ab})}) \circ (\text{id}_{m_a} \otimes \Lambda_{ba} - \text{id}_{\text{d}m_a \otimes m_{ab}}) \circ \\ &\quad \circ (\mathbf{a}_{m_a, A_a, m_{ab}} - \text{id}_{\text{d}m_a \otimes m_{ab}}) \circ (\Lambda_a \otimes \text{id}_{m_{ab}}) . \end{aligned} \quad (5.29)$$

As before, we have used our intuitive notation here.

5.4. Semistrict non-Abelian Deligne cohomology

Deligne cohomology describes gauge configurations on a principal bundle with connection modulo gauge transformations, which act simultaneously on the connection and the

transition functions of the bundle. Deligne cohomology for some categorified bundles was described previously: the case of Abelian gerbes was discussed in [36], the case of principal 2-bundles with strict structure 2-group was given in [37], and the case of principal 3-bundles was presented in [9], see also [38]. Here, we wish to give the full Deligne cohomology for principal 2-bundles with semistrict structure 2-group. For semistrict structure 2-group $B^2U(1)$, this reduces to ordinary, Abelian Deligne cohomology.

In the following, we consider a manifold X with cover $\mathfrak{U} = \{U_a\}$. We write $C^{p,q}(\mathfrak{U}, \mathfrak{a})$ for Čech p -cochains, which are defined on all non-empty intersections of $p + 1$ patches $U_{i_1} \cap \dots \cap U_{i_{p+1}}$ and which take values in the sheaf of \mathfrak{a} -valued q -forms.

Definition 5.17. *A degree- p Deligne cochain with values in a semistrict 2-group $N \rightrightarrows M$ with underlying 2-term L_∞ -algebra $\mathfrak{n} \rightarrow \mathfrak{m}$ consists of elements*

$$\begin{aligned} (n_{(1)}, \dots, n_{(p+1)}) &\in C^{p,0}(\mathfrak{U}, N) \times C^{p-1,1}(\mathfrak{U}, \mathfrak{n}) \times \dots \times C^{0,p}(\mathfrak{U}, \mathfrak{n}) , \\ (m_{(1)}, \dots, m_{(p)}) &\in C^{p-1,0}(\mathfrak{U}, M) \times C^{p-2,1}(\mathfrak{U}, \mathfrak{m}) \times \dots \times C^{0,p-1}(\mathfrak{U}, \mathfrak{m}) . \end{aligned} \quad (5.30)$$

A degree- p Deligne cochain with values in a semistrict 2-group consists therefore of $p + 1$ Čech cochains $n_{(k)}$ and p cochains $m_{(k)}$. The sum of the Čech and de Rham degrees of the $n_{(k)}$ is p , while it is $p - 1$ for the $m_{(k)}$. Compared to the analogous discussions of Deligne cochains for strict 2-groups in Schreiber & Waldorf [37], we dropped Čech cochains that are always cohomologous to trivial ones in the description of 2-bundles, cf. [9] and Proposition 3.15.

Using our results from the previous sections, we can describe Deligne cohomology with semistrict 2-groups up to degree 2.

Definition 5.18. *A degree- p Deligne cocycle with $p \leq 2$ is a degree- p Deligne cochain satisfying a cocycle relation. Explicitly, we have the following:*

- (i) *A degree-0 Deligne cocycle is an element $\{n_a\} \in C^{0,0}(\mathfrak{U}, N)$. The cocycle condition reads on $U_a \cap U_b$ as*

$$n_a = n_b . \quad (5.31)$$

- (ii) *A degree-1 Deligne cocycle consists of elements $\{n_{ab}\} \in C^{1,0}(\mathfrak{U}, N)$, $\{B_a\} \in C^{0,1}(\mathfrak{U}, \mathfrak{n})$ and $\{m_a\} \in C^{0,0}(\mathfrak{U}, M)$ such that*

$$n_{ab} : m_b \Rightarrow m_a \quad \text{and} \quad \text{id}_{m_a} + B_a : m_a \Rightarrow m_a . \quad (5.32)$$

The remaining cocycle conditions are¹⁵

$$\begin{aligned} n_{ab} \circ n_{bc} &= n_{ac} \quad \text{on} \quad U_a \cap U_b \cap U_c , \\ B_b &= \text{ad}_{n_{ab}}^\circ(B_a) + L_{n_{ab}}^*(dn_{ab}^{-1}) \quad \text{on} \quad U_a \cap U_b . \end{aligned} \quad (5.33)$$

¹⁵Here, $\text{ad}_{n_{ab}}^\circ$ is the differential of the map $n \mapsto n_{ab}^{-1} \circ n \circ n_{a,b}$ for some $n \in N$ with $\mathfrak{s}(n) = \mathfrak{t}(n) = \mathfrak{t}(n_{ab})$. Moreover, $L_{n_{ab}}^*$ denotes the pullback along the map $L_{n_{ab}} : n \mapsto n_{ab} \circ n$ for some $n \in N$ with $\mathfrak{s}(n) = \mathfrak{t}(n_{ab})$.

(iii) A degree-2 Deligne cocycle consists of elements $\{n_{abc}\} \in C^{2,0}(\mathfrak{U}, N)$, $\{\Lambda_{ab}\} \in C^{1,1}(\mathfrak{U}, \mathfrak{n})$ and $\{B_a\} \in C^{0,2}(\mathfrak{U}, \mathfrak{n})$ as well as $\{m_{ab}\} \in C^{1,0}(\mathfrak{U}, M)$ and $\{A_a\} \in C^{0,1}(\mathfrak{U}, \mathfrak{m})$ such that

$$\begin{aligned} n_{abc} : m_{ab} \otimes m_{bc} &\Rightarrow m_{ac} , \\ \Lambda_{ab} : A_b \otimes m_{ab} &\Rightarrow m_{ab} \otimes A_a - \mathrm{d}m_{ab} . \end{aligned} \quad (5.34a)$$

The remaining cocycle conditions are

$$n_{acd} \circ (n_{abc} \otimes \mathrm{id}_{m_{cd}}) \circ \mathbf{a}_{m_{ab}, m_{bc}, m_{cd}}^{-1} = n_{abd} \circ (\mathrm{id}_{m_{ab}} \otimes n_{bdc}) , \quad (5.34b)$$

$$\begin{aligned} \Lambda_{cb} \circ (\mathrm{id}_{A_b} \otimes n_{bac}) \circ \mathbf{a}_{A_b, m_{ba}, m_{ac}} &= \\ = (n_{bac} \otimes \mathrm{id}_{A_c} - \mathrm{d}n_{bac}) \circ [\mathbf{a}_{m_{ba}, m_{ac}, A_c}^{-1} - \mathrm{id}_{\mathrm{d}(m_{ba} \otimes m_{ac})}] \circ \\ \circ (\mathrm{id}_{m_{ba}} \otimes \Lambda_{ca} - \mathrm{id}_{\mathrm{d}m_{ba} \otimes m_{ac}}) \circ (\mathbf{a}_{m_{ba}, A_c, m_{ac}} - \mathrm{id}_{\mathrm{d}m_{ba} \otimes m_{ac}}) \circ (\Lambda_{ab} \otimes \mathrm{id}_{m_{ac}}) , \end{aligned} \quad (5.34c)$$

and

$$\begin{aligned} B_b \otimes \mathrm{id}_{m_{ab}} &= \mu(A_b, A_b, m_{ab}) + [\mathrm{id}_{m_{ab}} \otimes B_a + \mu(m_{ab}, A_a, A_a)] \circ \\ &\circ [-\mathrm{d}\Lambda_{ab} - \Lambda_{ab} \otimes \mathrm{id}_A - \mu(A_b, m_{ab}, A_a)] \circ \\ &\circ [-\mathrm{id}_{\mathrm{s}(\mathrm{d}\Lambda_{ab})} - \mathrm{id}_{A_b} \otimes (\Lambda_{ab} + \mathrm{id}_{\mathrm{d}m_{ab}})] . \end{aligned} \quad (5.34d)$$

on the relevant non-empty overlaps $U_a \cap U_b$, $U_a \cap U_b \cap U_c$ and $U_a \cap U_b \cap U_c \cap U_d$.

Definition 5.19. Any two degree- p Deligne cocycles are called cohomologous if and only if there is a degree- $(p-1)$ Deligne cochain relating both. In more detail, we define:

(i) Two degree-1 Deligne cocycles $(\{n_{ab}\}, \{B_a\}, \{m_a\})$ and $(\{\tilde{n}_{ab}\}, \{\tilde{B}_a\}, \{\tilde{m}_a\})$ are cohomologous if and only if there is a degree-0 Deligne cochain $\{n_a\} \in C^{0,0}(\mathfrak{U}, N)$ such that

$$\begin{aligned} n_a : m_a &\Rightarrow \tilde{m}_a \quad \text{on } U_a , \\ \tilde{n}_{ab} &= n_a \circ n_{ab} \circ n_b^{-1} \quad \text{on } U_a \cap U_b , \\ \tilde{B}_a &= \mathrm{ad}_{n_a}^\circ(B_a) + L_{n_a}^*(\mathrm{d}n_a^{-1}) \quad \text{on } U_a . \end{aligned} \quad (5.35)$$

(ii) Two degree-2 Deligne cocycles $(\{m_{ab}\}, \{n_{abc}\}, \{A_a\}, \{B_a\}, \{\Lambda_{ab}\})$ and $(\{\tilde{m}_{ab}\}, \{\tilde{n}_{abc}\}, \{\tilde{A}_a\}, \{\tilde{B}_a\}, \{\tilde{\Lambda}_{ab}\})$ are cohomologous if and only if there is a degree-1 Deligne cochain $(\{n_{ab}\}, \{\Lambda_a\}, \{m_a\})$ such that

$$\begin{aligned} n_{ab} : \tilde{m}_{ab} \otimes m_b &\Rightarrow m_a \otimes m_{ab} , \\ \Lambda_a : \tilde{A}_a \otimes m_a &\Rightarrow m_a \otimes A_a - \mathrm{d}m_a , \end{aligned} \quad (5.36a)$$

as well as

$$\begin{aligned} n_{ac} \circ (n_{abc} \otimes \text{id}_{m_c}) &= (\text{id}_{m_a} \otimes \tilde{n}_{abc}) \circ \mathbf{a}_{m_a, \tilde{m}_{ab}, \tilde{m}_{bc}} \circ (n_{ab} \otimes \text{id}_{\tilde{m}_{bc}}) \circ \\ &\circ \mathbf{a}_{m_{ab}, m_b, \tilde{m}_{bc}}^{-1} \circ (\text{id}_{m_{ab}} \otimes n_{bc}) \circ \mathbf{a}_{m_{ab}, m_{bc}, m_c} , \end{aligned} \quad (5.36b)$$

and

$$\begin{aligned} \tilde{B}_a \otimes \text{id}_{m_a} &= \mu(\tilde{A}_a, \tilde{A}_a, m_a) + [\text{id}_{m_a} \otimes B_a + \mu(m_a, A_a, A_a)] \circ \\ &\circ [-\text{d}\Lambda_a - \Lambda_a \otimes \text{id}_A - \mu(\tilde{A}_a, m_a, A_a)] \circ \\ &\circ [-\text{id}_{\mathfrak{s}(\text{d}\Lambda_a)} - \text{id}_{\tilde{A}_a} \otimes (\Lambda_a + \text{id}_{\text{d}m_a})] , \end{aligned} \quad (5.36c)$$

$$\begin{aligned} (\tilde{\Lambda}_{ba} \otimes \text{id}_{m_b}) \circ \mathbf{a}_{\tilde{A}_a, \tilde{m}_{ab}, m_b}^{-1} \circ (\text{id}_{\tilde{A}_a} \otimes n_{ab}) \circ \mathbf{a}_{\tilde{A}_a, m_a, m_{ab}} &= \\ = (\mathbf{a}_{\tilde{m}_{ab}, \tilde{A}_b, m_b}^{-1} - \text{id}_{\text{d}\tilde{m}_{ab} \otimes m_b}) \circ (\text{id}_{\tilde{m}_{ab}} \otimes \Lambda_b^{-1} - \text{id}_{\text{d}\tilde{m}_{ab} \otimes m_b}) \circ (\mathbf{a}_{m_{ab}, m_b, A_b} - \text{id}_{\text{d}(\tilde{m}_{ab} \otimes m_b)}) \circ \\ \circ (n_{ab}^{-1} \otimes \text{id}_{A_b} - \text{d}n_{ab}^{-1}) \circ (\mathbf{a}_{m_a, m_{ab}, A_b}^{-1} - \text{id}_{\text{d}(m_a \otimes m_{ab})}) \circ (\text{id}_{m_a} \otimes \Lambda_{ba} - \text{id}_{\text{d}m_a \otimes m_{ab}}) \circ \\ \circ (\mathbf{a}_{m_a, A_a, m_{ab}} - \text{id}_{\text{d}m_a \otimes m_{ab}}) \circ (\Lambda_a \otimes \text{id}_{m_{ab}}) . \end{aligned} \quad (5.36d)$$

Let us end this section by briefly commenting on the interpretation of elements of Deligne cohomology sets. The first case of degree-0 Deligne cocycles is readily understood. A degree-0 Deligne cocycles describes an N -valued function on X , which could be regarded as a principal 0-bundle.

The case of Deligne 1-cocycles is slightly more involved. If N is a group, which is the case, for instance, for strict Lie 2-groups, then a degree-1 Deligne cocycle defines a principal (1-)bundle with connection one-form B and a preferred section m . This data was called a crossed module bundle, from which crossed module bundle gerbes were constructed in [2], see also [39]. Recall that an Abelian bundle $(p+1)$ -gerbe over a manifold X can be constructed from the notion of an Abelian bundle p -gerbe, by considering a surjective submersion $Y \rightarrow X$ together with Abelian bundle p -gerbes over $Y^{[2]} := Y \times_X Y$. The analogous construction for crossed module bundle gerbes starts from a crossed module bundle. If N is not a group, then a Deligne 1-cocycle describes a 2-group principal bundle, which is a special form of a groupoid principal bundle. Considering 2-group principal bundles over $Y^{[2]}$ yields then to 2-group bundle gerbes or the principal 2-bundles described by Deligne 2-cocycles.

A degree-2 Deligne cocycle describes a semistrict principal 2-bundle with connective structure. Again, gauge equivalence is captured by the cohomology. To study such Deligne 2-cocycles further, it is useful to introduce the curvature forms of the connective structure; see also Proposition 5.3.

Definition 5.20. Let $(\{A_a\}, \{B_a\}, \{\Lambda_{ab}\})$ be a connective structure on a semistrict principal 2-bundle. The associated curvature forms are defined as follows:

$$\begin{aligned}\mathcal{F}_a &:= dA_a + A_a \otimes A_a + \mathfrak{s}(B_a) , \\ H_a &:= dB_a + \text{id}_{A_a} \otimes B_a - B_a \otimes \text{id}_{A_a} + \mu(A_a, A_a, A_a) .\end{aligned}\tag{5.37}$$

6. Application: Penrose–Ward transform

As an application of the theory of principal 2-bundles which we have developed in the previous sections, we now show how to generalise the results of [8]. Specifically, [8] established a Penrose–Ward transform that yields a bijection between holomorphic principal 2-bundles with strict structure 2-group over a twistor space and non-Abelian self-dual tensor fields on six-dimensional flat space-time. We can now replace the strict principal 2-bundles by semistrict ones in this construction.

In the following, we denote by \mathcal{O}_X the sheaf of holomorphic functions by Ω_X^p the sheaf of *holomorphic* differential p -forms on a complex (super)manifold X .

6.1. Supertwistor space

The twistor space P^6 underlying chiral six-dimensional field theories on flat complexified six-dimensional space-time \mathbb{C}^6 is the moduli space of α -planes or self-dual 3-planes in \mathbb{C}^6 . This twistor space has been described in great detail before [40, 10, 11], and its supersymmetric extension $P^{6|2n}$ was discussed in [8, 41, 9]. We therefore keep our following exposition brief.

The starting point is the chiral superspace $M^{6|8n} := \mathbb{C}^{6|8n}$. This space can be equipped with the coordinates (x^{AB}, η_I^A) , where $x^{AB} = -x^{BA}$ with $A, B, \dots = 1, \dots, 4$ are the usual Grassmann-even coordinates in spinor notation, η_I^A are the Grassmann-odd coordinates and $I, J, \dots = 1, \dots, 2n$ are the R-symmetry indices. We may raise and lower the spinor indices using the Levi-Civita symbol, that is, $x_{AB} = \frac{1}{2}\varepsilon_{ABCD}x^{CD} \Leftrightarrow x^{AB} = \frac{1}{2}\varepsilon^{ABCD}x_{CD}$. In the real setting, the R-symmetry group of the superconformal group $\text{OSp}(2, 6|2n)$ is

$$\text{Sp}(n) = \begin{cases} \text{Sp}(1) \cong \text{SU}(2) & \text{for } n = 1 \\ \text{Sp}(2) \cong \text{USp}(4) \subset \text{Sp}(4, \mathbb{C}) & \text{for } n = 2 \end{cases} .\tag{6.1}$$

The group $\text{Sp}(n)$ is defined as the elements of $\text{SU}(2n)$ leaving an antisymmetric $2n \times 2n$ matrix Ω invariant, which we can fix according to

$$\Omega = \text{diag}(\underbrace{\varepsilon, \dots, \varepsilon}_{n\text{-times}}) \quad \text{with} \quad \varepsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .\tag{6.2}$$

Working in the complex setting, we shall employ complexifications of the above groups.

We further introduce the superspace derivatives

$$P_{AB} := \frac{\partial}{\partial x^{AB}} \quad \text{and} \quad D_A^I := \frac{\partial}{\partial \eta_I^A} - 2\Omega^{IJ}\eta_J^B \frac{\partial}{\partial x^{AB}}, \quad (6.3)$$

which obey

$$\{D_A^I, D_B^J\} = -4\Omega^{IJ}P_{AB}. \quad (6.4)$$

Next, we let \mathbb{P}^3 be the complex projective 3-space and define $F^{9|8n} := \mathbb{C}^{6|8n} \times \mathbb{P}^3$, which we call the correspondence space. It can be equipped with coordinates $(x^{AB}, \eta_I^A, \lambda_A)$ where λ_A are homogeneous coordinates on \mathbb{P}^3 . On the correspondence space, we introduce the twistor distribution, denoted by D , which is the integrable distribution of rank $3|6n$ that is generated by the vector fields

$$D := \text{span}\{V^A, V^{IAB}\} \quad \text{with} \quad V^A := \lambda_B \partial^{AB} \quad \text{and} \quad V^{IAB} := \frac{1}{2}\varepsilon^{ABCD}\lambda_C D_D^I. \quad (6.5)$$

The supertwistor space $P^{6|2n}$ is then defined to be the associated leaf space $P^{6|2n} := F^{9|8n}/D$. We can now establish a twistor correspondence which is captured by the double fibration

$$\begin{array}{ccc} & F^{9|8n} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ P^{6|2n} & & M^{6|8n} \end{array} \quad (6.6)$$

where π_2 is the trivial projection while

$$\pi_1 : (x^{AB}, \eta_I^A, \lambda_A) \mapsto (z^A, \eta_I, \lambda_A) = ((x^{AB} + \Omega^{IJ}\eta_I^A \eta_J^B)\lambda_B, \eta_I^A \lambda_A, \lambda_A), \quad (6.7)$$

contains the so-called incidence relation

$$z^A = (x^{AB} + \Omega^{IJ}\eta_I^A \eta_J^B)\lambda_B \quad \text{and} \quad \eta_I = \eta_I^A \lambda_A. \quad (6.8)$$

This incidence relation yields a geometric correspondence between points $x \in M^{6|8n}$ and complex projective 3-spaces $\hat{x} = \pi_1(\pi_2^{-1}(x)) \hookrightarrow P^{6|2n}$ as well as points $p \in P^{6|2n}$ in twistor space and totally null $3|6n$ -superplanes $\pi_2(\pi_1^{-1}(p)) \hookrightarrow M^{6|8n}$. It also follows that $P^{6|2n}$ the quadric hypersurface given by the zero locus

$$z^A \lambda_A - \Omega^{IJ} \eta_I \eta_J = 0 \quad (6.9)$$

inside the total space of the holomorphic fibration $\mathbb{C}^{4|2n} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow \mathbb{P}^3$ with fibre coordinates z^A and η_I as well as base coordinates λ_A .

Remark 6.1. *In our subsequent discussion, we shall always choose the standard Stein cover $\hat{\mathcal{U}} = \{\hat{U}_a\}$ on the twistor space $P^{6|2n} \rightarrow \mathbb{P}^3$ (generated by the standard Stein cover on \mathbb{P}^3) and the induced cover $\mathcal{U}' := \{U'_a := \pi_1^{-1}(U_a)\}$ on the correspondence space $F^{9|8n}$, respectively.*

6.2. Penrose–Ward transform

To formulate the Penrose–Ward transform, we first need to introduce a few basic notions. In particular, we will need the sheaf of relative differential p -forms, denoted by $\Omega_{\pi_1}^p$, on $F^{9|8n}$ along the fibration $\pi_1 : F^{9|8n} \rightarrow P^{6|2n}$. It is defined by the short exact sequence

$$0 \longrightarrow \pi_1^* \Omega_{P^{6|2n}}^1 \wedge \Omega_{F^{9|8n}}^{p-1} \longrightarrow \Omega_{F^{9|8n}}^p \longrightarrow \Omega_{\pi_1}^p \longrightarrow 0. \quad (6.10)$$

In addition, if $\text{pr}_{\pi_1} : \Omega_{F^{9|8n}}^p \rightarrow \Omega_{\pi_1}^p$ denotes the quotient mapping, we can define the relative exterior derivative d_{π_1} by setting

$$d_{\pi_1} := \text{pr}_{\pi_1} \circ d : \Omega_{\pi_1}^p \rightarrow \Omega_{\pi_1}^{p+1}, \quad (6.11)$$

where d denotes the usual (holomorphic) exterior derivative on the correspondence space. In local coordinates $(x^{AB}, \eta_I^A \lambda_A)$ on $F^{9|8n}$, the relative exterior derivative is presented in terms of the vector fields of the twistor distribution (6.5). It characterises the so-called relative holomorphic de Rham complex, which is the complex that is given in terms of an injective resolution of the topological inverse $\pi_1^{-1} \mathcal{O}_{P^{6|2n}}$ of the sheaf $\mathcal{O}_{P^{6|2n}}$ on the correspondence space $F^{9|8n}$:

$$0 \longrightarrow \pi_1^{-1} \mathcal{O}_{P^{6|2n}} \longrightarrow \mathcal{O}_{F^{9|8n}} \xrightarrow{d_{\pi_1}} \Omega_{\pi_1}^1 \xrightarrow{d_{\pi_1}} \Omega_{\pi_1}^2 \xrightarrow{d_{\pi_1}} \dots. \quad (6.12)$$

Note that $\pi_1^{-1} \mathcal{O}_{P^{6|2n}}$ consists of those holomorphic functions that are locally constant along the fibres of $\pi_1 : F^{9|8n} \rightarrow P^{6|2n}$.

Next, let Φ' be a holomorphic semistrict principal 2-bundles on the correspondence space $F^{9|8n}$, with $\mathcal{B}\mathcal{G} = (\{e\}, M, N)$ as its semistrict structure 2-group. Let us denote the semistrict Lie 2-algebra associated with $\mathcal{B}\mathcal{G}$ by $\mathcal{L} = (\mathfrak{m}, \mathfrak{n})$. The bundle Φ' is described by holomorphic $\mathcal{B}\mathcal{G}$ -valued transition functions $(\{m'_{ab}\}, \{n'_{abc}\})$ relative to the cover \mathcal{U}' .

As we shall see momentarily, the Penrose–Ward transform will be based on so-called relative degree-2 Deligne cohomology. For this reason, we wish to equip Φ' with a relative connective structure and study its behaviour and equivalence transformations. Concretely, Φ' is then described by a degree-2 Deligne cocycle

$$(\{m'_{ab}\}, \{n'_{abc}\}, \{\Lambda'_{ab}\}, \{A'_a\}, \{B'_a\}) \quad (6.13)$$

with $\{m'_{ab}\} \in C_{\pi_1}^{1,0}(\mathcal{U}', M)$, $\{n'_{abc}\} \in C_{\pi_1}^{2,0}(\mathcal{U}', N)$, $\{\Lambda'_{ab}\} \in C_{\pi_1}^{1,1}(\mathcal{U}', \mathfrak{n})$, $\{A'_a\} \in C_{\pi_1}^{0,1}(\mathcal{U}', \mathfrak{m})$, and $\{B'_a\} \in C_{\pi_1}^{0,2}(\mathcal{U}', \mathfrak{n})$. Here, the subscript ' π_1 ' indicates that these are relative differential forms. For instance, the Λ'_{ab} and A'_a take values in $\Omega_{\pi_1}^1 \otimes \mathfrak{n}$ and $\Omega_{\pi_1}^1 \otimes \mathfrak{m}$, respectively, while the B'_a take values in $\Omega_{\pi_1}^2 \otimes \mathfrak{n}$. In addition, we call the relative connective structure flat if

and only if the corresponding curvatures vanish

$$\begin{aligned}\mathcal{F}'_a &:= d_{\pi_1} A'_a + \frac{1}{2} \mu_2(A'_a, A'_a) - \mu_1(B'_a) = 0, \\ H'_a &:= d_{\pi_1} B'_a + \mu_2(A'_a, B'_a) - \frac{1}{3!} \mu_3(A'_a, A'_a, A'_a) = 0.\end{aligned}\tag{6.14}$$

The final ingredient we shall need is a holomorphic semistrict principal 2-bundle $\hat{\Phi}$ on $P^{6|2n}$ with $\mathcal{B}\mathcal{G} = (\{e\}, M, N)$ as its semistrict structure 2-group. The bundle $\hat{\Phi}$ is described by holomorphic $\mathcal{B}\mathcal{G}$ -valued transition functions $(\{\hat{m}_{ab}\}, \{\hat{n}_{abc}\})$ relative to the cover $\hat{\mathcal{U}}$. Following Manin [42], $\hat{\Phi}$ will be called $M^{6|8n}$ -trivial if and only if it is holomorphically trivial on $\hat{x} = \pi_1(\pi_2^{-1}(x)) \hookrightarrow P^{6|2n}$ for all $x \in M^{6|8n}$; see also Definition 3.19.

Then we have the following result.

Proposition 6.2. *Consider $\pi_1 : F^{9|8n} \rightarrow P^{6|2n}$; see (6.6). There is a bijection between*

- (i) *equivalence classes of topologically trivial $M^{6|8n}$ -trivial holomorphic semistrict principal 2-bundles on $P^{6|2n}$ and*
- (ii) *equivalence classes of holomorphically trivial semistrict principal 2-bundles on $F^{9|8n}$ equipped with a relative connective structure which is globally flat.*

Proof. (i) \rightarrow (ii) Let $\hat{\Phi}$ be an $M^{6|8n}$ -trivial holomorphic semistrict principal 2-bundle on the twistor space $P^{6|2n}$ described by holomorphic transition functions $(\{\hat{m}_{ab}\}, \{\hat{n}_{abc}\})$. Furthermore, let $\Phi' = \pi_1^* \hat{\Phi}$ be its pullback to the correspondence space $F^{9|8n}$; see also Definition 3.18. It is described by holomorphic transition functions $(\{m'_{ab}\}, \{n'_{abc}\})$ which are annihilated by the relative exterior derivative d_{π_1} . In particular, it is described by the relative degree-2 Deligne cocycle

$$(\{m'_{ab} = \pi_1^* \hat{m}_{ab}\}, \{n'_{abc} = \pi_1^* \hat{n}_{abc}\}, \{\Lambda'_{ab} = 0\}, \{A'_a = 0\}, \{B'_a = 0\}). \tag{6.15}$$

Since $\hat{\Phi}$ is $M^{6|8n}$ -trivial, its pullback Φ' is holomorphically trivial on all of $F^{9|8n}$. Therefore, there exists a relative degree-2 Deligne cochain relating the degree-2 Deligne cocycle (6.15) to the cocycle

$$(\{m''_{ab} = \text{id}_{e_a}\}, \{n''_{abc} = \text{id}_{\text{id}_{e_a}}\}, \{\Lambda''_{ab} \neq 0\}, \{A''_a \neq 0\}, \{B''_a \neq 0\}). \tag{6.16}$$

From (5.34) and (5.25), we realise that $\Lambda''_{ab} : A''_b \Rightarrow A''_a$ and

$$\Lambda''_{ac} = \Lambda''_{ab} \circ \Lambda''_{bc}. \tag{6.17}$$

Defining

$$\Xi_{ab} := \Lambda''_a - \text{id}_{A''_a}, \tag{6.18}$$

we immediately see that

$$\Xi_{ac} = \Xi_{ab} + \Xi_{bc} , \quad (6.19)$$

that is, $\{\Xi_{ab}\}$ (and, correspondingly $\{\Lambda_a''\}$) is a representative of an element in the Abelian Čech cohomology group $H^1(F^{9|8n}, \Omega_{\pi_1}^1 \otimes \mathfrak{n})$. This cohomology group, however, vanishes as was demonstrated in [10, 8] (see also [11]). Therefore, we have a splitting

$$\Xi_{ab} = \Xi_a - \Xi_b \quad \text{with} \quad \Xi_a : A_a''' - A_a'' \Rightarrow 0 , \quad (6.20)$$

where the A_a''' define a globally defined \mathfrak{m} -valued relative 1-form $A_{\pi_1}''' \in H^0(F^{9|8n}, \Omega_{\pi_1}^1 \otimes \mathfrak{m})$, that is, $A_a''' = A_{\pi_1}'''|_{U_a'}$ and $A_a''' = A_b'''$ on $U_a' \cap U_b'$. Equivalently, we may write

$$\Lambda_{ab}'' = \Lambda_a'' \circ (\Lambda_b'')^{-1} \quad \text{with} \quad \Lambda_a'' : A_a''' \Rightarrow A_a'' \quad \text{and} \quad \Lambda_a'' := \Omega_a + \text{id}_{A_a''} . \quad (6.21)$$

Thus, using (5.36d) with Λ_a'' , we see that the degree-2 Deligne cocycle (6.16) is cohomologous to

$$(\{m_{ab}''' = \text{id}_{e_a}\}, \{n_{abc}''' = \text{id}_{\text{id}_{e_a}}\}, \{\Lambda_{ab}''' = 0\}, \{A_a''' \neq 0\}, \{B_a''' \neq 0\}) , \quad (6.22)$$

where the B_a''' yield a globally defined \mathfrak{n} -valued relative 2-form $B_{\pi_1}''' \in H^0(F^{9|8n}, \Omega_{\pi_1}^2 \otimes \mathfrak{n})$, that is, $B_a''' = B_{\pi_1}'''|_{U_a'}$ and $B_a''' = B_b'''$ on $U_a' \cap U_b'$.

Altogether, we have obtained a holomorphically trivial semistrict principal 2-bundle Φ' on the correspondence space, equipped with a globally defined relative connective structure represented by (A_{π_1}, B_{π_2}) . As this relative connective structure is pure gauge, its curvatures necessarily vanish, and, therefore, the relative connective structure is flat.

(ii) \rightarrow (i) Conversely, starting from a holomorphically trivial semistrict principal 2-bundle Φ' on the correspondence space represented by a relative degree-2 Deligne cocycle of the form (6.22) with a relative connective structure that is flat, we can use a generalised Poincaré lemma [43] to find a relative degree-2 Deligne cochain to transform (6.22) into a cocycle of the form (6.16). This cocycle descends down to twistor space to a relative degree-2 Deligne cocycle of the form (6.15). \square

Note that there are equivalence transformations acting on the ingredients of this construction. For instance, constructing the degree-2 Deligne cochains explicitly that mediate between the different degree-2 Deligne cocycles amounts to solving Riemann–Hilbert problems whose solutions are not unique.

Next, we write the relative exterior derivative explicitly as

$$d_{\pi_1} = e_A V^A + e_{IAB} V^{IAB} = e_{[A} \lambda_{B]} \partial^{AB} + e_I^{AB} \lambda_A D_B^I , \quad (6.23)$$

thereby introducing the relative 1-forms e_A and $e_{IAB} = \frac{1}{2} \varepsilon_{ABCD} e_I^{CD}$ which are defined dually to V^A and V^{IAB} . Notice that since $\lambda_A V^A = \lambda_A V^{IAB} = 0$, these differential 1-forms are defined modulo terms proportional to λ_A ; see also [10, 8] for more details.

Lemma 6.3. *Let $\alpha_{\pi_1} \in H^0(F^{9|8n}, \Omega_{\pi_1}^1)$, $\beta_{\pi_1} \in H^0(F^{9|8n}, \Omega_{\pi_1}^2)$, and $\gamma_{\pi_1} \in H^0(F^{9|8n}, \Omega_{\pi_1}^3)$. These relative differential forms are then expanded in λ_A according to*

$$\begin{aligned}
\alpha_{\pi_1} &= e_{[A} \lambda_{B]} \alpha^{AB} + e_I^{AB} \lambda_A \alpha_B^I, \\
\beta_{\pi_1} &= -\frac{1}{4} e_A \wedge e_B \lambda_C \varepsilon^{ABCD} \beta_D^E \lambda_E + \frac{1}{2} e_A \lambda_B \wedge e_I^{EF} \lambda_E \varepsilon^{ABCD} \beta_{CD}^I + \\
&\quad + \frac{1}{2} e_I^{CA} \lambda_C \wedge e_J^{DB} \lambda_D \beta_{AB}^{IJ}, \\
\gamma_{\pi_1} &= -\frac{1}{3} e_A \wedge e_B \wedge e_C \lambda_D \varepsilon^{ABCD} \gamma^{EF} \lambda_E \lambda_F + \\
&\quad - \frac{1}{4} e_A \wedge e_B \lambda_C \varepsilon^{ABCD} \wedge e_I^{EF} \lambda_E (\gamma_D^{GI}{}_F)_0 \lambda_G + \\
&\quad + \frac{1}{4} e_A \lambda_B \wedge e_I^{EF} \lambda_E \wedge e_J^{GH} \lambda_G \varepsilon^{ABCD} (\gamma_{CD}^{IJ}{}_{FH})_0 + \\
&\quad + \frac{1}{6} e_I^{DA} \lambda_D \wedge e_J^{EB} \lambda_E \wedge e_K^{FC} \lambda_F \gamma_{ABC}^{IJK},
\end{aligned} \tag{6.24}$$

where the coefficient functions depend only on the superspace coordinates $(x^{AB}, \eta_I^A) \in M^{6|8n}$. The component $(\gamma_A^{BI}{}_C)_0$ is the totally trace-less part of $\gamma_A^{BI}{}_C$ while $(\gamma_{AB}^{IJ}{}_{CD})_0$ denotes the part of $\gamma_{AB}^{IJ}{}_{CD}$ that vanishes under contraction with ε^{ABCD} .

Proof. This is a direct consequence of the explicit form of direct images of the sheaves $\Omega_{\pi_1}^1$ and $\Omega_{\pi_1}^2$ under the projection $\pi_2 : F^{9|8n} \rightarrow M^{6|8n}$. See references [10, 8] for a detailed derivation. \square

Remark 6.4. *Note that differential 1-, 2- and 3-forms α , β , and γ on chiral superspace $M^{6|8n}$ have components*

$$(\alpha_{AB}, \alpha_B^I), \quad (\beta_A^B, \beta_{ABC}^I, \beta_{AB}^{IJ}), \quad \text{and} \quad (\gamma_{AB}, \gamma^{AB}, \gamma_A^{BI}{}_C, \gamma_{AB}^{IJ}{}_{CD}, \gamma_{ABC}^{IJK}), \tag{6.25}$$

where $\gamma_A^{BI}{}_C$ is traceless over the AB indices. By virtue of Lemma 6.3, we realise that all of these components for the 1- and 2-forms and some of these components for the 3-form appear in the expansion of relative 1-, 2- and 3-forms α_{π_1} , β_{π_1} , and γ_{π_1} on the correspondence $F^{9|8n}$. Note further that the components $(\gamma_{AB}, \gamma^{AB})$ represent the self-dual and anti-self dual parts of a Graßmann-even differential 3-form γ on $M^{6|0}$.

These considerations then enable us to prove the following Penrose–Ward transform.

Theorem 6.5. *Consider the double fibration (6.6). There is a bijection between*

- (i) *equivalence classes of topologically trivial $M^{6|8n}$ -trivial holomorphic semistrict principal 2-bundles on $P^{6|2n}$ and*
- (ii) *gauge equivalence classes of (complex holomorphic) solutions to the constraint equations*

$$\mathcal{F}_A^B = 0, \quad \mathcal{F}_{ABC}^I = 0, \quad \text{and} \quad \mathcal{F}_{AB}^{IJ} = 0, \tag{6.26a}$$

and

$$\begin{aligned}
H^{AB} &= 0 , \\
H_A^{BI} &= \delta_C^B \psi_A^I - \frac{1}{4} \delta_A^B \psi_C^I , \\
H_{AB\dot{C}D}^{IJ} &= \varepsilon_{ABCD} \phi^{IJ} , \\
H_{ABC}^{IJK} &= 0
\end{aligned} \tag{6.26b}$$

on chiral superspace $M^{6|8n}$. Here, the curvatures read explicitly as

$$\begin{aligned}
\mathcal{F}_A^B &= \partial^{BC} A_{CA} - \partial_{CA} A^{BC} + \mu_2(A^{BC}, A_{CA}) - \mu_1(B_A^B) , \\
\mathcal{F}_{AB\dot{C}}^I &= \partial_{AB} A_{\dot{C}}^I - D_{\dot{C}}^I A_{AB} + \mu_2(A_{AB}, A_{\dot{C}}^I) - \mu_1(B_{AB\dot{C}}^I) , \\
\mathcal{F}_{AB}^{IJ} &= D_A^I A_B^J + D_B^J A_A^I + \mu_2(A_A^I, A_B^J) + 4\Omega^{IJ} A_{AB} - \mu_1(B_{AB}^{IJ})
\end{aligned} \tag{6.27a}$$

and

$$\begin{aligned}
H_{AB} &= \nabla_{C(A} B_{B)}^C + \mu_3(A_{C(A}, A^{CD}, A_{B)D}) , \\
H^{AB} &= \nabla^{C(A} B_{C}^{B)} + \mu_3(A^{C(A}, A_{CD}, A^{B)D}) , \\
H_A^{BI} &= \nabla_C^I B_A^B - \nabla^{DB} B_{DA}^I + \nabla_{DA} B^{DBI}_C - \mu_3(A_C^I, A^{BD}, A_{DA}) , \\
H_{AB\dot{C}D}^{IJ} &= \nabla_{AB} B_{\dot{C}D}^{IJ} - \nabla_C^I B_{AB\dot{D}}^J - \nabla_D^J B_{AB\dot{C}}^I - \\
&\quad - 2\Omega^{IJ} (\varepsilon_{ABF[C} B_{D]}^F - \varepsilon_{CDF[A} B_{B]}^F) - \mu_3(A_{AB}, A_{\dot{C}}^I, A_{\dot{D}}^J) , \\
H_{ABC}^{IJK} &= \nabla_A^I B_{BC}^{JK} + \nabla_B^J B_{AC}^{IK} + \nabla_C^K B_{AB}^{IJ} + \\
&\quad + 4\Omega^{IJ} B_{ABC}^K + 4\Omega^{IK} B_{ACB}^J + 4\Omega^{JK} B_{BCA}^I - \mu_3(A_A^I, A_B^J, A_C^K) .
\end{aligned} \tag{6.27b}$$

Before proving the theorem, let us make a few comments. The fields ψ_A^I are Graßmann-odd spinor fields while the fields ϕ^{IJ} are Graßmann-even scalar fields. The condition $H^{AB} = 0$ implies that the Graßmann-even part of the 3-form H is self-dual; see Remark 6.4. Altogether, $(H_{AB}, \psi_A^I, \phi^{IJ})$ constitutes an $\mathcal{N} = (n, 0)$ tensor multiplet for $n = 0, 1, 2$. Note that only for $n = 2$, the condition $\phi^{IJ} \Omega_{IJ} = 0$ arises, so that we always find the correct number of scalar fields. See also Saemann & Wolf [8–10] for more details on this discussion.

Proof of theorem: (i) \rightarrow (ii) By virtue of Proposition 6.2, topologically trivial $M^{6|8n}$ -trivial holomorphic semistrict principal 2-bundles on twistor space correspond to holomorphic-semistrict principal 2-bundles on $F^{9|8n}$ equipped with a relative connective structure which is globally flat and vice versa. Therefore, such a bundle on twistor space yields a globally defined relative connective structure $(A_{\pi_1}, B_{\pi_1}) \in H^0(F^{9|8n}, \Omega_{\pi_1}^1 \otimes \mathfrak{m}) \times H^0(F^{9|8n}, \Omega_{\pi_1}^2 \otimes \mathfrak{n})$ on the correspondence space which is flat, that is,

$$\begin{aligned}
\mathcal{F}_{\pi_1} &= d_{\pi_1} A_{\pi_1} + \frac{1}{2} \mu_2(A_{\pi_1}, A_{\pi_1}) - \mu_1(B_{\pi_1}) = 0 , \\
H_{\pi_1} &= d_{\pi_1} B_{\pi_1} + \mu_2(A_{\pi_1}, B_{\pi_1}) - \frac{1}{3!} \mu_3(A_{\pi_1}, A_{\pi_1}, A_{\pi_1}) = 0 .
\end{aligned} \tag{6.28}$$

Upon using (6.23) and the expansions given in Lemma 6.3, we arrive at the constraint equations (6.26) and (6.27) after a few algebraic manipulations.

(ii) \rightarrow (i) The converse is also readily derived. Given a solution to (6.26) and (6.27), by Lemma 6.3 we can always construct a globally defined relative connective structure $(A_{\pi_1}, B_{\pi_1}) \in H^0(F^{9|8n}, \Omega_{\pi_1}^1 \otimes \mathfrak{m}) \times H^0(F^{9|8n}, \Omega_{\pi_1}^2 \otimes \mathfrak{n})$ on the correspondence space which is flat. This defines a holomorphically trivial semistrict principal 2-bundles on $F^{9|8n}$ equipped with a flat relative connective structure. The construction of a topologically trivial $M^{6|8n}$ -trivial holomorphic semistrict principal 2-bundles on twistor space then follows directly from Proposition 6.2. \square

Remark 6.6. *Finally, we would like to mention that the gauge transformations of the connective structure $(A_{AB}, A_A^I, B_A^B, B_{AB}^I, B_{AB}^{IJ})$ on $M^{6|8n}$ follow directly from the large class of equivalence relations between relative Deligne 2-cocycles of the form (6.22) on $F^{9|8n}$. The Deligne 1-cochains parametrising the equivalence relations between relative Deligne 2-cocycles of the form (6.22) are expressed in terms of $p \in H^0(F^{9|8n}, M)$ and $\Lambda_{\pi_1} \in H^0(F^{9|8n}, \Omega_{\pi_1}^1 \otimes \mathfrak{n})$. Their λ_A -expansions read as*

$$p = p(x, \eta) \quad \text{and} \quad \Lambda_{\pi_1} = e_{[A} \lambda_{B]} \Lambda^{AB}(x, \eta) + e_I^{AB} \lambda_A \Lambda_B^I(x, \eta) . \quad (6.29)$$

Such Deligne 1-cochains are therefore described by $p(x, \eta)$, $\Lambda_{AB}(x, \eta)$, and $\Lambda_A^I(x, \eta)$ which themselves form a Deligne 1-cochain encoding an equivalence relation between Deligne 2-cocycles on the chiral superspace $M^{6|8n}$. The gauge transformations are then simply of the form given in Proposition 5.11.

Appendix

A. Strong homotopy Lie algebras

In this appendix, we recall the definitions of strong homotopy Lie algebras and their Chevalley–Eilenberg algebras as well as the homotopy Maurer–Cartan equations together with their infinitesimal gauge symmetries.

Recall that a permutation σ of $i + j$ elements is called an (i, j) -unshuffle, if the first i and the last j images of σ are ordered: $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i + 1) < \dots < \sigma(i + j)$. Moreover, the graded Koszul sign $\chi(\sigma; x_1, \dots, x_n)$ of elements x_i of a graded vector space is defined via the equation

$$x_1 \wedge \dots \wedge x_n = \chi(\sigma; x_1, \dots, x_n) x_{\sigma(1)} \wedge \dots \wedge x_{\sigma(n)} \quad (\text{A.1})$$

in the free graded algebra $\wedge(x_1, \dots, x_n)$, where \wedge is considered graded antisymmetric.

Definition A.1. [44] An L_∞ -algebra or *strong homotopy Lie algebra* is a non-positively-graded vector space $L = \oplus_{p \leq 0} L_p$ endowed with n -ary multilinear totally antisymmetric products μ_n , $n \in \mathbb{N}^*$, of degree $2 - n$, that satisfy the homotopy Jacobi identities

$$\sum_{i+j=n} \sum_{\sigma} \chi(\sigma; x_1, \dots, x_n) (-1)^{i \cdot j} \mu_{i+1}(\mu_j(x_{\sigma(1)}, \dots, x_{\sigma(j)}), x_{\sigma(j+1)}, \dots, x_{\sigma(i+j)}) = 0 \quad (\text{A.2})$$

for all $n \in \mathbb{N}^*$, where the sum over σ is taken over all (i, j) unshuffles.

An alternative sign convention is given in [45], which is obtained from the above one by inverting the signs of all elements of L . The homotopy Jacobi identities (A.2) then read as

$$\sum_{i+j=n} \sum_{\sigma} \chi(\sigma; x_1, \dots, x_n) \mu_{j+1}(\mu_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(i+j)}) = 0. \quad (\text{A.3})$$

A simple example of an L_∞ -algebra is a differential graded Lie algebra, for which μ_1 is the differential, μ_2 is the Lie bracket and $\mu_i = 0$ for $i \geq 3$. Another example of an L_∞ -algebra is given by the 2-term L_∞ -algebras of Definition 2.39.

Definition A.2. A \mathbb{Z} -graded coalgebra is a \mathbb{Z} -graded vector space $L = \oplus_{p \in \mathbb{Z}} L_p$ endowed with a *coproduct*. That is, we have a map $\Delta : A \rightarrow A \otimes A$ of degree 0 such that $(\mathbb{1} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathbb{1}) \circ \Delta$. A *coderivation* of degree k on a coalgebra C is a linear map $D : C \rightarrow C$ of degree k such that $\Delta \circ D = (\mathbb{1} \otimes D + D \otimes \mathbb{1}) \circ \Delta$. A *differential graded coalgebra* is a graded coalgebra endowed with a coderivation D of degree 1 such that $D^2 = 0$.

Each L_∞ -algebra yields naturally a differential graded coalgebra. We start from an L_∞ -algebra L , and shift the degree of each element by -1 , arriving at $L[-1]$. The symmetric tensor algebra $\odot^\bullet L[-1]$ of $L[-1]$ can be regarded as a graded coalgebra with coproduct

$$\Delta(\ell_1 \odot \dots \odot \ell_n) := \sum_{i=0}^n \sum_{\sigma} (\ell_{\sigma(1)} \otimes \dots \otimes \ell_{\sigma(i)}) \otimes (\ell_{\sigma(i+1)} \otimes \dots \otimes \ell_{\sigma(n)}) , \quad (\text{A.4})$$

where the sum over σ is taken over all $(i, n-i)$ unshuffles. Note that on $L[-1]$, the higher products μ_n have degree 1 and we can add them to a differential D , which acts as μ_i on $L[-1]^{\otimes i}$ and on higher tensor powers of $L[-1]$ as a coderivation. The property $D^2 = 0$ is then equivalent to the homotopy Jacobi identities.

On the other hand, given a differential graded coalgebra with negative grading, we can derive a corresponding L_∞ -algebra. Altogether, we arrive at the following proposition.

Proposition A.3. An L_∞ -algebra is equivalent to a differential graded coalgebra with negative grading.

Instead of working with coalgebras, it is usually more convenient to work directly with differential graded algebras. Assuming that the vector subspaces $L_p \subset L$ are finite dimensional, we can consider the dual complex $L[-1]^*$ to $L[-1]$.

Definition A.4. The Chevalley–Eilenberg algebra of an L_∞ -algebra L is the dual of the differential graded coalgebra $\odot^\bullet L[-1]$. In particular, $\mathbf{CE}(L) := \odot^\bullet(L[-1]^*)$ and the differential $d_{\mathbf{CE}} = D^*$ is the adjoint of the differential D in $\odot^\bullet L[-1]$.

It is straightforward to verify the $\mathbf{CE}(L)$ is indeed a differential graded algebra.

The Chevalley–Eilenberg algebra of a Lie algebra \mathfrak{g} is a differential graded algebra that encodes the Lie bracket via the equation

$$d_{\mathbf{CE}} \tilde{\tau}^k + \frac{1}{2} f_{ij}^k \tilde{\tau}^i \otimes \tilde{\tau}^j = 0, \quad (\text{A.5})$$

where the $\tilde{\tau}^k$ form a basis of \mathfrak{g}^* and f_{ij}^k are the structure constants of \mathfrak{g} with respect to the dual basis τ_k , $\langle \tilde{\tau}^k, \tau_j \rangle = \delta_j^k$. Evaluated at an element $a \in \mathfrak{g}[-1]$, we have

$$d_{\mathbf{CE}} a + \frac{1}{2} [a, a] = 0, \quad (\text{A.6})$$

the Maurer–Cartan equation of the differential graded algebra. This equation can be generalised to the case of L_∞ -algebras.

Definition A.5. An element ϕ of an L_∞ -algebra is called a homotopy Maurer–Cartan element if it satisfies the homotopy Maurer–Cartan equations

$$\sum_{i=1}^{\infty} \frac{(-1)^{i(i+1)/2}}{i!} \mu_i(\phi, \dots, \phi) = 0. \quad (\text{A.7})$$

Theorem A.6. The homotopy Maurer–Cartan equations are invariant under the following infinitesimal symmetries parameterised by an element $\lambda \in L_0$:

$$\phi \rightarrow \phi + \delta\phi \quad \text{with} \quad \delta\phi = \sum_i \frac{(-1)^{i(i-1)/2}}{(i-1)!} \mu_i(\lambda, \phi, \dots, \phi). \quad (\text{A.8})$$

Proof. The general proof of this theorem is found e.g. in [46]. Here, we give a shortened version for the case $\phi \in L_1$, which is the one we are interested in. We start by computing the homotopy Jacobi identities (A.2) for the tuple $(\lambda, \phi, \dots, \phi)$, obtaining

$$\begin{aligned} & \sum_{i+j=n} \binom{n-1}{j-1} (-1)^{ij} \mu_{i+1}(\mu_j(\lambda, \phi, \dots, \phi), \phi, \dots, \phi) + \\ & + \sum_{i+j=n, i \geq 1} \binom{n-1}{j} (-1)^{ij+n-1} \mu_{i+1}(\mu_j(\phi, \dots, \phi), \phi, \dots, \phi, \lambda) = 0 \end{aligned} \quad (\text{A.9})$$

or

$$\begin{aligned} \sum_{i+j=n} \frac{1}{(j-1)!i!} (-1)^{1+i(n-i)-\frac{n}{2}+\frac{n^2}{2}} \mu_{i+1}(\mu_j(\phi, \dots, \phi, \lambda), \phi, \dots, \phi) + \\ + \sum_{i+j=n, i \geq 1} \frac{(-1)^{1+i(n-i)+n-1-\frac{n}{2}+\frac{n^2}{2}}}{j!(i-1)!} \mu_{i+1}(\mu_j(\phi, \dots, \phi), \phi, \dots, \phi, \lambda) = 0. \end{aligned} \quad (\text{A.10})$$

Next, we note the following identities for $i + j = n$:

$$\begin{aligned} (-1)^{1+i(n-i)+n-1-\frac{n}{2}+\frac{n^2}{2}} &= (-1)^{i(n-i)+\frac{n}{2}+\frac{n^2}{2}} = (-1)^{i(i+1)/2+j(j+1)/2}, \\ (-1)^{((i+1)(i+2)+j(j-1))/2} &= (-1)^{1+2i+i^2-\frac{n}{2}-in+\frac{n^2}{2}} = (-1)^{1+i(n-i)-\frac{n}{2}+\frac{n^2}{2}}. \end{aligned} \quad (\text{A.11})$$

Now we can compute the variation of (A.7) under the transformations (A.8):

$$\begin{aligned} \delta \left(\sum_{i=1}^{\infty} \frac{(-1)^{i(i+1)/2}}{i!} \mu_i(\phi, \dots, \phi) \right) &= \sum_{i=1}^{\infty} \frac{(-1)^{i(i+1)/2}}{(i-1)!} \mu_i(\delta\phi, \dots, \phi) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{(i(i+1)+j(j-1))/2}}{(i-1)!(j-1)!} \mu_i(\mu_j(\lambda, \phi, \dots, \phi), \phi, \dots, \phi) \\ &= \sum_{n=1}^{\infty} \sum_{i+j=n} \frac{(-1)^{((i+1)(i+2)+j(j-1))/2}}{i!(j-1)!} \mu_{i+1}(\mu_j(\lambda, \phi, \dots, \phi), \phi, \dots, \phi) \\ &= - \sum_{n=1}^{\infty} \sum_{i+j=n, i \geq 1} \frac{(-1)^{i(i+1)/2+j(j+1)/2}}{j!(i-1)!} \mu_{i+1}(\mu_j(\phi, \dots, \phi), \phi, \dots, \phi, \lambda) \\ &= - \sum_{i=1}^{\infty} \frac{(-1)^{i(i+1)/2}}{(i-1)!} \mu_{i+1} \left(\sum_{j=1}^{\infty} \frac{(-1)^{j(j+1)/2}}{j!} \mu_j(\phi, \dots, \phi), \phi, \dots, \phi, \lambda \right) \\ &= 0 \end{aligned} \quad (\text{A.12})$$

as a consequence of the homotopy Maurer–Cartan equations (A.7). \square

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